SOLUTION

CE 325 Spring 2026 HW#7

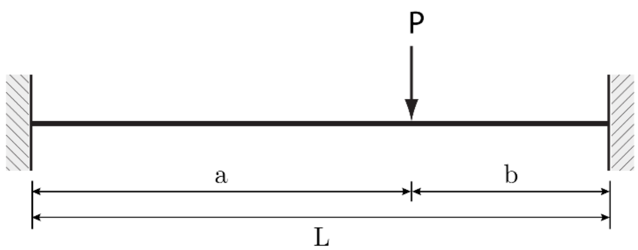
Due Thursday, April 09 by 1:30pm ET

1. (10 pts) Derive the 4th column of the 4x4 beam element stiffness matrix [k]:

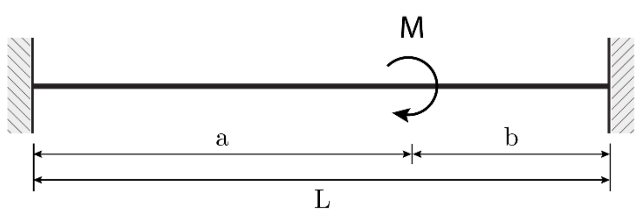
$$[k] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

2. Derive the fixed end forces/moments for the following beam loading cases:

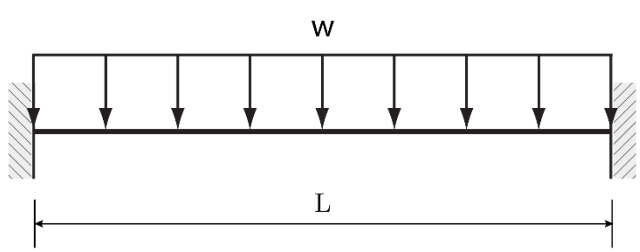
a. (5 pts)



b. (5 pts)



c. (5 pts)



Problem 1

Executive Summary

Problem Statement

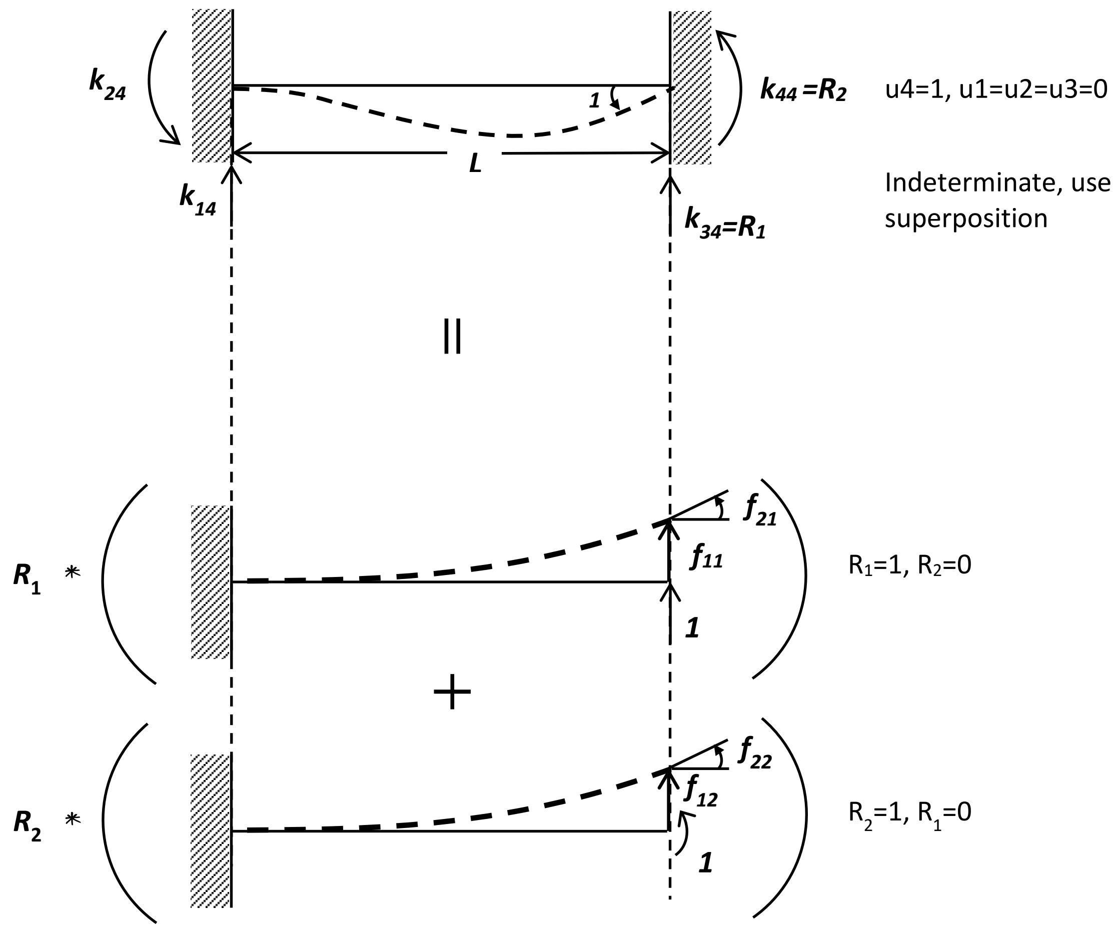
Derive the 4th column of the 4x4 beam element stiffness matrix [k]

Results

$$[k] = \frac{\boldsymbol{EI}}{\boldsymbol{L^3}}\begin{bmatrix} 12 & 6L & -12 & \boldsymbol{6L} \\ 6L & 4L^2 & -6L & \boldsymbol{2L^2} \\ -12 & -6L & 12 & \boldsymbol{-6L} \\ 6L & 2L^2 & -6L & \boldsymbol{4L^2} \end{bmatrix}$$

Problem 1

Technical Summary



Compatibility

Displacement $u3 = 0 = f_{11}R_1 + f_{12}R_2$

Rotation $u4 = 1 = f_{21}R_1 + f_{22}R_2$

Derive flexibilities from moment-curvature equation

$f_{11} = \frac{L^3}{3EI}$ $f_{12} = \frac{L^2}{2EI}$

$f_{21} = \frac{L^2}{2EI}$ $f_{22} = \frac{L}{EI}$

Substitute Values

$$0 = \frac{L^3}{3EI} R_1 + \frac{L^2}{2EI} R_2$$

$$1 = \frac{L^2}{2EI} R_1 + \frac{L}{EI} R_2$$

$$\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{bmatrix} \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

Solve System of Equations

$$R_1 = k_{34} = -\frac{6EI}{L^2}$$

$$R_2 = k_{44} = \frac{4EI}{L}$$

From Equilibrium

$$\Sigma F_y = 0 \rightarrow k_{14} = -k_{34}$$

$$k_{14} = \frac{6EI}{L^2}$$

$$\Sigma M_B = 0 \rightarrow k_{24} = -k_{34}L - k_{44}$$

$$k_{24} = \frac{2EI}{L}$$

4th Column of Beam Element Stiffness Matrix

$$\{\boldsymbol{k_{i4}}\} = \begin{Bmatrix} \frac{6EI}{L^2} \\ \frac{2EI}{L} \\ -\frac{6EI}{L^2} \\ \frac{4EI}{L} \end{Bmatrix}$$

Problem 2

Executive Summary

Problem Statement

Derive the fixed end forces/moments for the beam loading cases.

Results

Concentrated Load

$$\{\boldsymbol{Q_f}\} = \begin{Bmatrix} \frac{Pb^2}{L^3}(3a+b) \\ \frac{Pab^2}{L^2} \\ \frac{Pa^2}{L^3}(a+3b) \\ -\frac{Pa^2b}{L^2} \end{Bmatrix}$$

Concentrated Moment

$$\{\boldsymbol{Q_f}\} = \begin{Bmatrix} -\frac{6Mab}{L^3} \\ \frac{Mb}{L^2}(b-2a) \\ \frac{6Mab}{L^3} \\ \frac{Ma}{L^2}(a-2b) \end{Bmatrix}$$

Uniformly Distributed Load

$$\{\boldsymbol{Q_f}\} = \begin{Bmatrix} \frac{wL}{2} \\ \frac{wL^2}{12} \\ \frac{wL}{2} \\ -\frac{wL^2}{12} \end{Bmatrix}$$

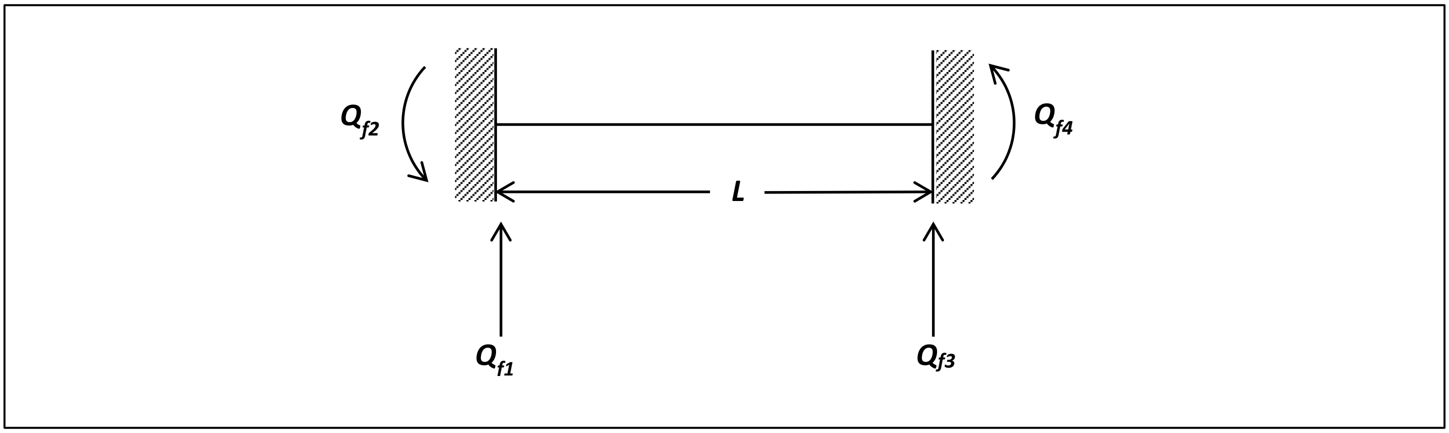
Uniformly Distributed Moment

$$\{\boldsymbol{Q_f}\} = \begin{Bmatrix} -m \\ 0 \\ m \\ 0 \end{Bmatrix}$$

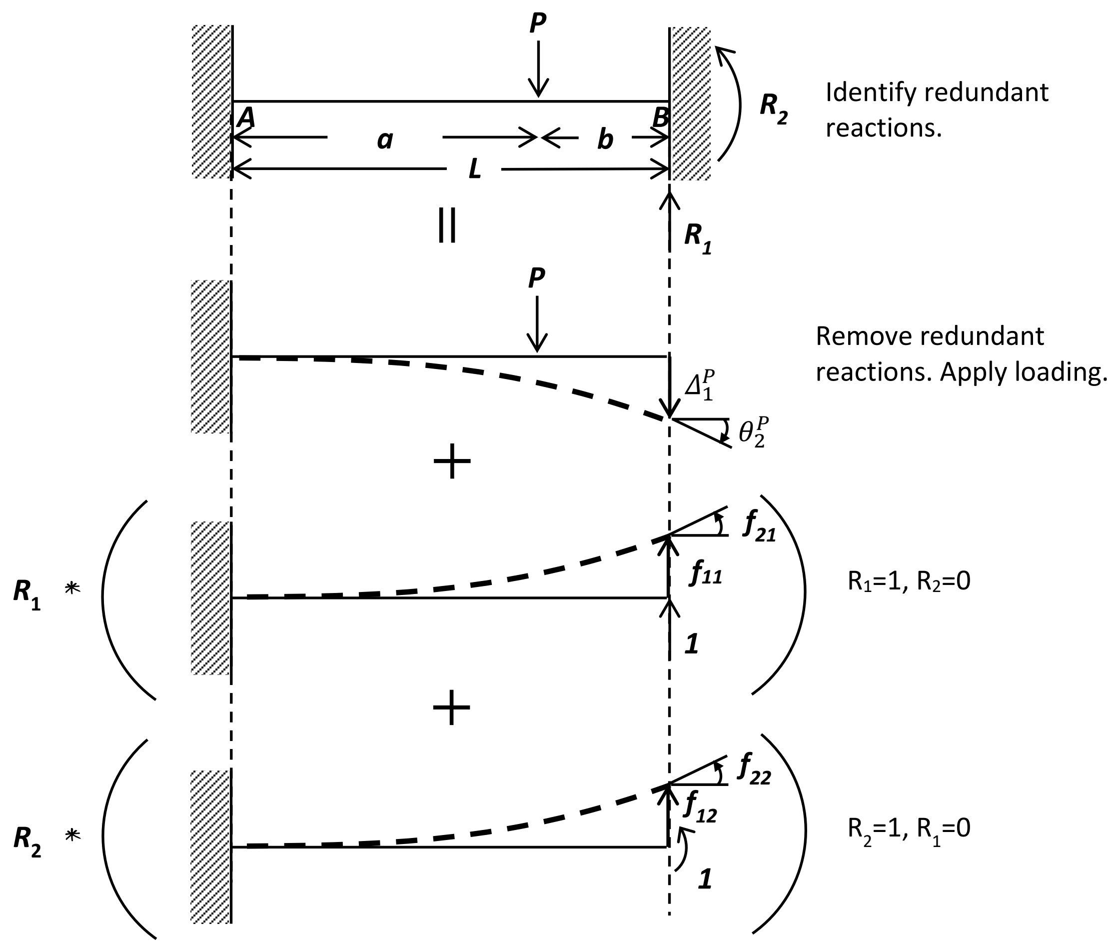
Problem 2

Technical Summary

Fixed End Force/Moment Notation



a) Concentrated Load



Compatibility

Displacement $0 = \Delta_1^P + f_{11}R_1 + f_{12}R_2$

Rotation $0 = \theta_1^P + f_{21}R_1 + f_{22}R_2$

Recall

$f_{11} = \frac{L^3}{3EI}$ $f_{12} = \frac{L^2}{2EI}$

$f_{21} = \frac{L^2}{2EI}$ $f_{22} = \frac{L}{EI}$

Loading Deflections

$\Delta_1^P = -\frac{Pa^2}{6EI}(3L - a)$

$\theta_2^P = -\frac{Pa^2}{2EI}$

Substitute Values

$$0 = -\frac{Pa^2}{6EI}(3L - a) + \frac{L^3}{3EI}R_1 + \frac{L^2}{2EI}R_2$$

$$0 = -\frac{Pa^2}{2EI} + \frac{L^2}{2EI}R_1 + \frac{L}{EI}R_2$$

$$\begin{Bmatrix} \frac{Pa^2}{6EI}(3L - a) \\ \frac{Pa^2}{2EI} \end{Bmatrix} = \begin{bmatrix} \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

Solve system of Equations

$$R_1 = Q_{f3} = \frac{Pa^2}{L^3}(a + 3b)$$

$$R_2 = Q_{f4} = -\frac{Pa^2 b}{L^2}$$

From Equilibrium

$$\Sigma F_y = 0 \rightarrow Q_{f1} = P - Q_{f3}$$

$$Q_{f1} = \frac{Pb^2}{L^3}(3a + b)$$

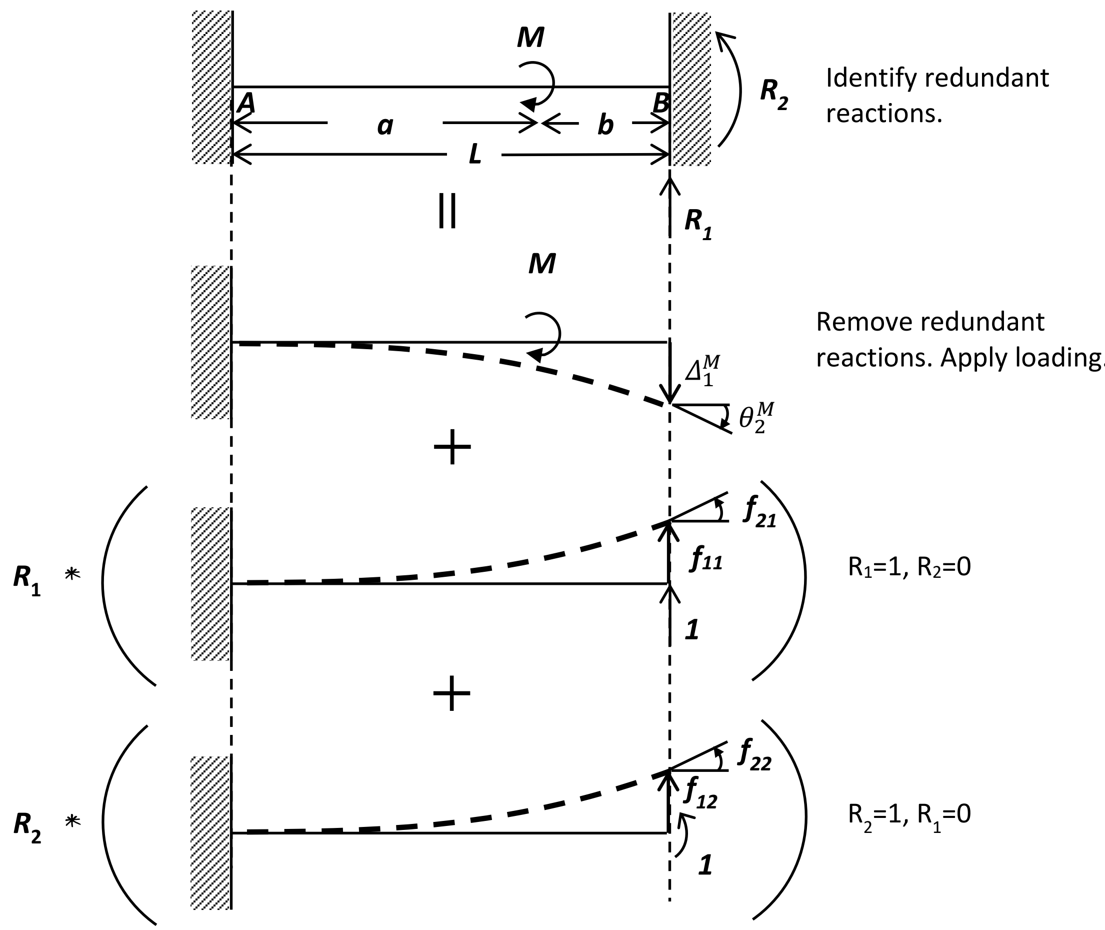
$$\Sigma M_B = 0 \rightarrow Q_{f2} = Pa - Q_{f3}L - Q_{f4}$$

$$Q_{f2} = \frac{Pab^2}{L^2}$$

Fixed End Forces/Moments

$$\{\boldsymbol{Q_f}\} = \begin{Bmatrix} \frac{\boldsymbol{Pb^2}}{\boldsymbol{L^3}}(\boldsymbol{3a} + \boldsymbol{b}) \\ \frac{\boldsymbol{Pab^2}}{\boldsymbol{L^2}} \\ \frac{\boldsymbol{Pa^2}}{\boldsymbol{L^3}}(\boldsymbol{a} + \boldsymbol{3b}) \\ -\frac{\boldsymbol{Pa^2b}}{\boldsymbol{L^2}} \end{Bmatrix}$$

b) Concentrated Moment



Compatibility

Displacement $0 = \Delta_1^M + f_{11}R_1 + f_{12}R_2$

Rotation $0 = \theta_1^M + f_{21}R_1 + f_{22}R_2$

Recall

$f_{11} = \frac{L^3}{3EI}$ $\quad f_{12} = \frac{L^2}{2EI}$

$f_{21} = \frac{L^2}{2EI}$ $\quad f_{22} = \frac{L}{EI}$

Loading Deflections

$\Delta_1^M = -\frac{M}{2EI}(a^2 + 2ab)$

$\theta_2^M = -\frac{Ma}{EI}$

Substitute Values

$$0 = -\frac{M}{2EI}(a^2 + 2ab) + \frac{L^3}{3EI}R_1 + \frac{L^2}{2EI}R_2$$

$$0 = -\frac{Ma}{EI} + \frac{L^2}{2EI}R_1 + \frac{L}{EI}R_2$$

$$\begin{Bmatrix} \frac{M}{2EI}(a^2 + 2ab) \\ \frac{Ma}{EI} \end{Bmatrix} = \begin{bmatrix} \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

Solve system of Equations

$$R_1 = Q_{f3} = \frac{6Mab}{L^3}$$

$$R_2 = Q_{f4} = \frac{Ma}{L^2}(a - 2b)$$

From Equilibrium

$$\Sigma F_y = 0 \rightarrow Q_{f1} = -Q_{f3}$$

$$Q_{f1} = -\frac{6Mab}{L^3}$$

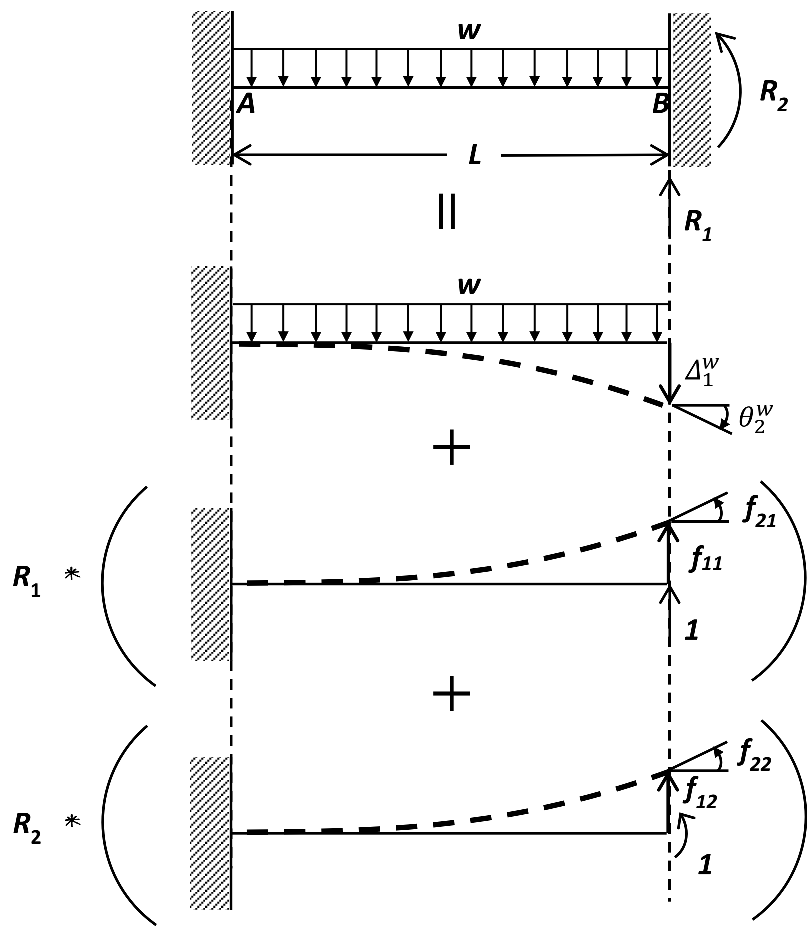
$$\Sigma M_B = 0 \rightarrow Q_{f2} = M - Q_{f3}L - Q_{f4}$$

$$Q_{f2} = \frac{Mb}{L^2}(b - 2a)$$

Fixed End Forces/Moments

$$\{\boldsymbol{Q_f}\} = \begin{Bmatrix} -\frac{\boldsymbol{6Mab}}{\boldsymbol{L^3}} \\ \frac{\boldsymbol{Mb}}{\boldsymbol{L^2}}(\boldsymbol{b - 2a}) \\ \frac{\boldsymbol{6Mab}}{\boldsymbol{L^3}} \\ \frac{\boldsymbol{Ma}}{\boldsymbol{L^2}}(\boldsymbol{a - 2b}) \end{Bmatrix}$$

c) Uniformly Distributed Load



Identify redundant reactions.

Remove redundant reactions. Apply loading.

R_1=1, R_2=0

R_2=1, R_1=0

Compatibility

Displacement $\quad 0 = \Delta_1^w + f_{11}R_1 + f_{12}R_2$

Rotation $\quad 0 = \theta_1^W + f_{21}R_1 + f_{22}R_2$

Recall

$f_{11} = \frac{L^3}{3EI} \qquad f_{12} = \frac{L^2}{2EI}$

$f_{21} = \frac{L^2}{2EI} \qquad f_{22} = \frac{L}{EI}$

Loading Deflections

$\Delta_1^w = -\frac{wL^4}{8EI}$

$\theta_2^w = -\frac{wL^3}{6EI}$

Substitute Values

$$0 = -\frac{wL^4}{8EI} + \frac{L^3}{3EI}R_1 + \frac{L^2}{2EI}R_2$$

$$0 = -\frac{wL^3}{6EI} + \frac{L^2}{2EI}R_1 + \frac{L}{EI}R_2$$

$$\begin{Bmatrix} \frac{wL^4}{8EI} \\ \frac{wL^3}{6EI} \end{Bmatrix} = \begin{bmatrix} \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

Solve system of Equations

$$R_1 = Q_{f3} = \frac{wL}{2}$$

$$R_2 = Q_{f4} = -\frac{wL^2}{12}$$

From Equilibrium

$$\Sigma F_y = 0 \rightarrow Q_{f1} = wL - Q_{f3}$$

$$Q_{f1} = \frac{wL}{2}$$

$$\Sigma M_B = 0 \rightarrow Q_{f2} = \frac{wL^2}{2} - Q_{f3}L - Q_{f4}$$

$$Q_{f2} = \frac{wL^2}{12}$$

Fixed End Forces/Moments

$$\{\boldsymbol{Q_f}\} = \begin{Bmatrix} \frac{\boldsymbol{wL}}{\boldsymbol{2}} \\ \frac{\boldsymbol{wL^2}}{\boldsymbol{12}} \\ \frac{\boldsymbol{wL}}{\boldsymbol{2}} \\ -\frac{\boldsymbol{wL^2}}{\boldsymbol{12}} \end{Bmatrix}$$