1d.

$\bar{u}_x = \sum_{i=0}^{n} a_i x^i$, n=1 (linear) $\qquad \bar{u}_x = a_0 + a_1 x$

$@x = 0,\ \bar{u}_x = u_1 \qquad \therefore \qquad a_0 = u_1$

$@x = L,\ \bar{u}_x = u_3 \qquad \therefore \qquad a_1 = \frac{u_3 - u_1}{L} \qquad \bar{u}_x = u_1 + \frac{u_3 - u_1}{L} x$

$$\bar{u}_x = \left(1 - \frac{x}{L}\right) u_1 + \left(\frac{x}{L}\right) u_3$$

Similarly:

$$\bar{u}_y = \left(1 - \frac{x}{L}\right) u_2 + \left(\frac{x}{L}\right) u_4$$

$$\bar{u} = [N]\{u\} \qquad \begin{Bmatrix} \bar{u}_x \\ \bar{u}_y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\varepsilon_{axial} = \frac{d\bar{u}_x}{dx} = \begin{bmatrix} \frac{d}{dx} & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_x \\ \bar{u}_y \end{Bmatrix} = \boldsymbol{D}[N]\{u\} = [B]\{u\}$$

$$[B] = \begin{bmatrix} \frac{dN_1}{dx} & 0 & \frac{dN_3}{dx} & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} & 0 & \frac{1}{L} & 0 \end{bmatrix}$$

$$[k]_{FEM} = \int_V B^T E B\, dV$$

*Note: to save space 2x2 [k] derived for uniaxial element; planar truss element has same terms embedded in 4x4 matrix padded with zeros

$$[k]_{FEM} = \int_V \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} \begin{Bmatrix} -\frac{1}{L} & \frac{1}{L} \end{Bmatrix} E\,dV = E \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} \int_0^L \int_A dA\,dx = \frac{E}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^L A\,dx$$

$$= \frac{EA}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^L dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2a.

$$A(x) = A_b\left(1 - \frac{x}{2L}\right) \quad [k]_{exact} = \frac{EA_b}{2L\ln(0.5)} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \approx 0.721 \frac{EA_b}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2b.

$$[k]_{linear} = \int_V \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} \begin{Bmatrix} -\frac{1}{L} & \frac{1}{L} \end{Bmatrix} E\,dV = \frac{E}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^L A(x)\,dx$$

$$= \frac{EA_b}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^L \left(1 - \frac{x}{2L}\right) dx = \frac{EA_b}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left. \left(x - \frac{x^2}{4L}\right) \right|_0^L = 0.75 \frac{EA_b}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2c.

$\bar{u}_x = \sum_{i=0}^{n} a_i x^i$, n=2 (quadratic) $\qquad \bar{u}_x = a_0 + a_1 x + a_2 x^2$

$a_1 \equiv \frac{u_2 - u_1}{2L}$ **Note: prescribed constant satisfies boundary conditions

$@x = 0,\ \bar{u}_x = u_1 \qquad \therefore \qquad a_0 = u_1$

$$@x = L,\ \bar{u}_x = u_2 \qquad \therefore \qquad a_2 = \frac{u_2 - u_1}{2L^2}$$

$$\bar{u}_x = u_1 + \left(\frac{u_2 - u_1}{2L}\right)x + \left(\frac{u_2 - u_1}{2L^2}\right)x^2 \qquad \bar{u}_x = \left(1 - \frac{x}{2L} - \frac{x^2}{2L^2}\right)u_1 + \left(\frac{x}{2L} + \frac{x^2}{2L^2}\right)u_2$$

$$\left[N\right] = \left[\begin{array}{cc} 1 - \frac{x}{2L} - \frac{x^2}{2L^2} & \frac{x}{2L} + \frac{x^2}{2L^2} \end{array}\right] \qquad \left[B\right] = \left[\begin{array}{cc} -\frac{1}{2L} - \frac{x}{L^2} & \frac{1}{2L} + \frac{x}{L^2} \end{array}\right]$$

$$\left[k\right]_{quadratic} = \int_V \left\{\begin{array}{c} -\frac{1}{2L} - \frac{x}{L^2} \\ \frac{1}{2L} + \frac{x}{L^2} \end{array}\right\}\left\{\begin{array}{cc} -\frac{1}{2L} - \frac{x}{L^2} & \frac{1}{2L} + \frac{x}{L^2} \end{array}\right\} E\,dV = \frac{E}{L^4}\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right]\int_0^L \left(x^2 + xL + \frac{L^2}{4}\right)A\left(x\right)dx$$

$$= \frac{EA_b}{L^4}\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right]\int_0^L \left(x^2 + xL + \frac{L^2}{4}\right)\left(1 - \frac{x}{2L}\right)dx = \frac{EA_b}{L^4}\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right]\Bigg|_0^L \left(-\frac{x^4}{8L} + \frac{x^3}{6} + \frac{7x^2L}{16} + \frac{xL^2}{4}\right)$$

$$= \frac{35EA_b}{48L}\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right] \approx 0.729\frac{EA_b}{L}\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right]$$

***Note: converges from a stiffer solution (0.75 $_{\text{linear}}$ to 0.729 $_{\text{quadratic}}$ to 0.721 $_{\text{exact}}$)