September 24, 2024
Matrix Displacement Method
Beams (BEBT)
Part I - formulation, [k]

Bernoulli - Euler Beam Theory (BEBT)

Assumptions : Isotropic/homogeneous, initially straight, vertical axis of symmetry, prismatic

slender (Length $\geq$ 10 * depth, $L/d \geq 10$)

$\sigma_{bending} = \dfrac{-My}{I}$ M = bending moment

y = distance from neutral axis

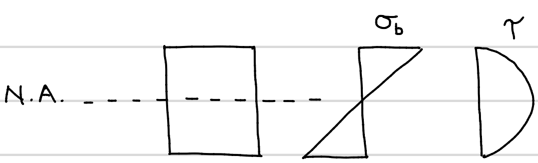
I = moment of inertia (2nd moment of area $\int y^2 \, dA$)

$\tau_{shear} = \dfrac{VQ}{Ib}$ V = shear force

Q = 1st moment of area $\int y \, dA$

b = width

Neutral Axis , i.e. y-direction centroid $\bar{y} = \dfrac{\int y \, dA}{A}$
(N.A.)



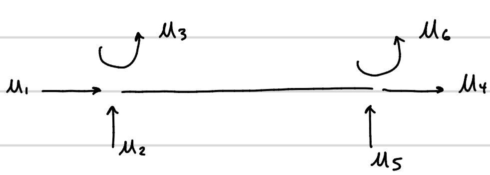
Uniaxial

axial displacements

* Beam

transverse displacements and rotations

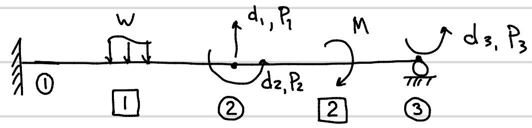
2D Frame
(coming soon)



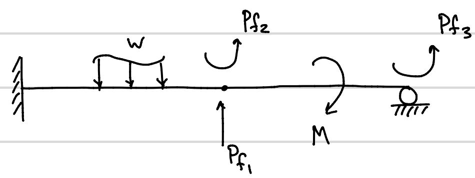
axial + transverse displacements and rotations

MDM Formulation for Beams

Structural-level Superposition



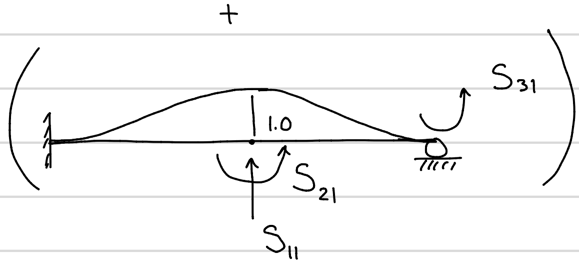
$=$



Apply member loading
$d_1 = d_2 = d_3 = 0$

$+$

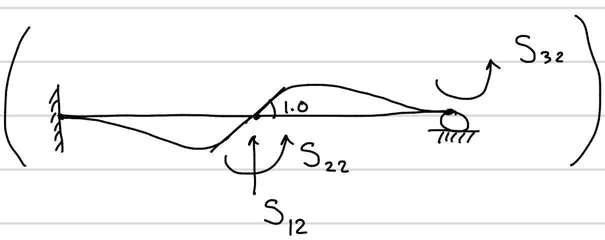
$d_1 *$



$d_1 = 1 \quad d_2 = d_3 = 0$

$+$

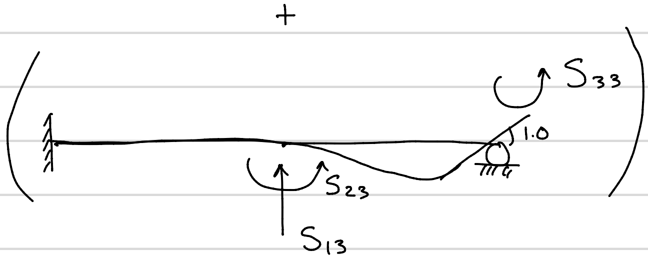
$d_2 *$



$d_2 = 1.0 \ , \ d_1 = d_3 = 0$

$+$

$d_3 *$

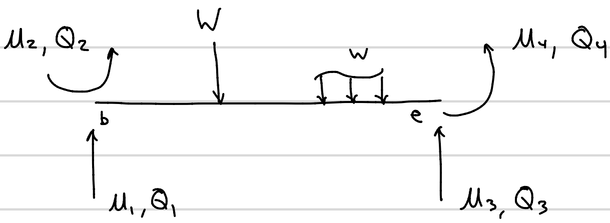


$d_3 = 1.0 \ , \ d_1 = d_2 = 0$

$$P_1 = P_{f_1} + S_{11} d_1 + S_{12} d_2 + S_{13} d_3$$
$$P_2 = P_{f_2} + S_{21} d_1 + S_{22} d_2 + S_{23} d_3$$
$$P_3 = P_{f_3} + S_{31} d_1 + S_{32} d_2 + S_{33} d_3$$

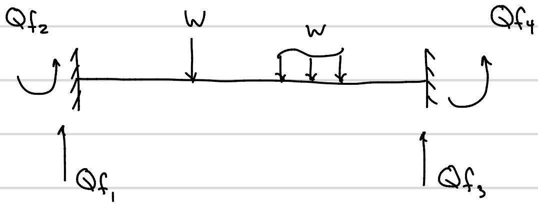
$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} P_{f_1} \\ P_{f_2} \\ P_{f_3} \end{Bmatrix} + \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

Member-level * Note: similar to uniaxial members, no difference in local/global coordinates for beams



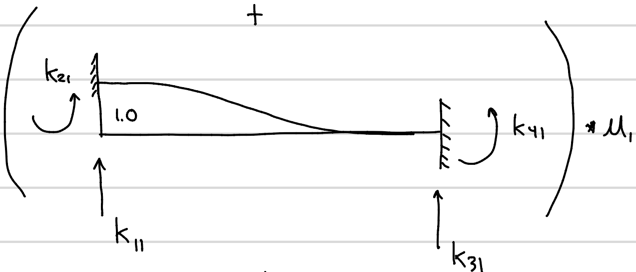
Superposition

=



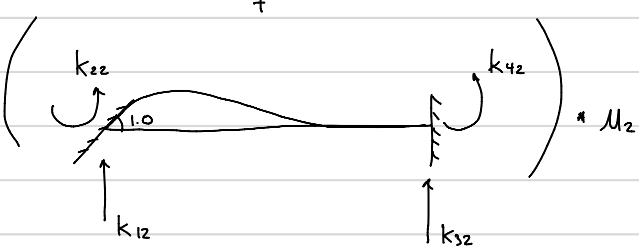
Apply loading
$u_1 = u_2 = u_3 = u_4 = 0$

\+

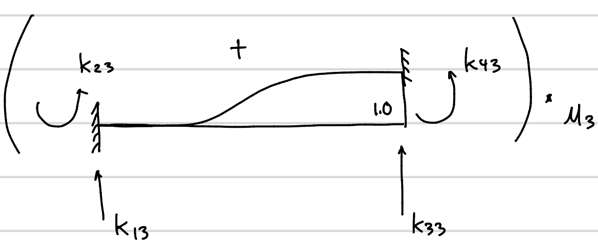


$u_1 = 1 \quad u_2 = u_3 = u_4 = 0$

\+

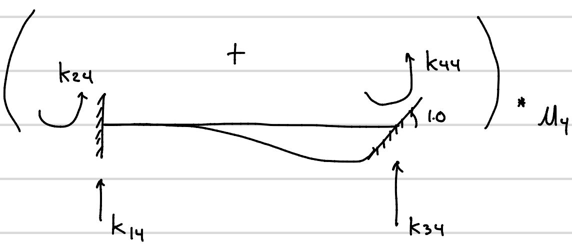


$u_2 = 1 \quad u_1 = u_3 = u_4 = 0$



$u_3 = 1, \quad u_1 = u_2 = u_4 = 0$

$u_4 = 1, \quad u_1 = u_2 = u_3 = 0$

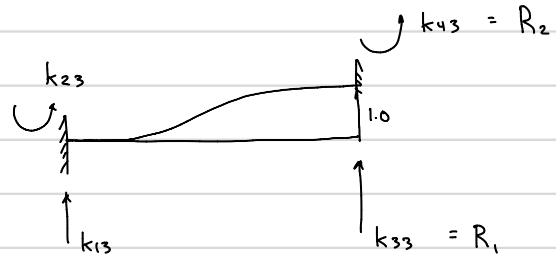


$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} Qf_1 \\ Qf_2 \\ Qf_3 \\ Qf_4 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\{Q\} = \{Qf\} + [k]\{u\}$$

Need to derive stiffness terms (direct approach)

<u>3rd</u> column of $[k]$ $\underline{u_3} = 1.0$, $u_1 = u_2 = u_4 = 0$



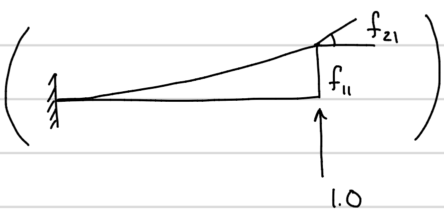
indeterminate, use superposition (flexibility approach)

Uniaxial $P = \left(\dfrac{EA}{L}\right) \delta$

stiffness (k)

$=$

R_1 *



$R_1 = 1.0$, $R_2 = 0$

$\delta = \left(\dfrac{L}{EA}\right) P$

flexibility $\dfrac{1}{k} \equiv f$

R_2 *

$R_2 = 1.0$, $R_1 = 0$

* Stiffness superposition apply displacements force equilibrium

<u>Compatibility</u>

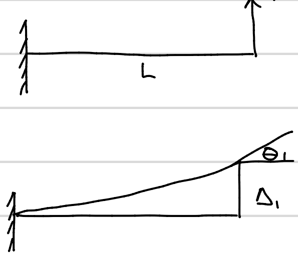
* Flexibility superposition apply forces displacement compatibility

$u_3 = \Delta_1 + \Delta_2 = 1.0$ $\quad 1.0 = f_{11} R_1 + f_{12} R_2$

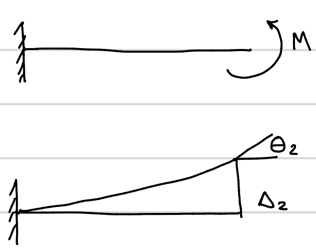
$u_4 = \theta_1 + \theta_2 = 0$ $\quad 0 = f_{21} R_1 + f_{22} R_2$

$$\begin{Bmatrix} 1.0 \\ 0 \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

R_1



R_2



Δ_i, θ_i can be derived from integration of moment-curvature relation
$$EI\,v'' = M(x)$$

$$\Delta_1 = \frac{PL^3}{3EI} \qquad \theta_1 = \frac{PL^2}{2EI} \qquad \Delta_2 = \frac{ML^2}{2EI} \qquad \theta_2 = \frac{ML}{EI}$$

$$f_{11} = \frac{L^3}{3EI} \qquad f_{21} = \frac{L^2}{2EI} \qquad f_{12} = \frac{L^2}{2EI} \qquad f_{22} = \frac{L}{EI}$$

$$\begin{Bmatrix} 1.0 \\ 0 \end{Bmatrix} = \begin{bmatrix} \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

Solving system of equations
$$R_1 = \frac{12EI}{L^3} = k_{33}$$
$$R_2 = \frac{-6EI}{L^2} = k_{43}$$

From element equilibrium
$$\Sigma F_y = 0 \qquad k_{13} + k_{33} = 0 \qquad k_{13} = \frac{-12EI}{L^3}$$
$$\Sigma M_b = 0 \qquad k_{23} + k_{43} + k_{33} * L = 0 \qquad k_{23} = \frac{-6EI}{L^2}$$

* Following the same flexibility superposition procedure for the 1st, 2nd, and 4th DOFs :

$$[k] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Now we need Qf ?