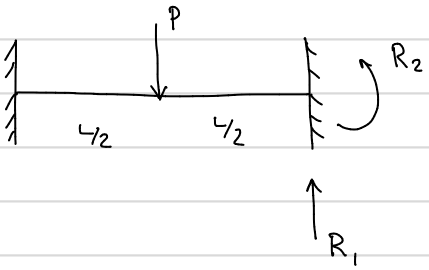
September 26, 2024
Matrix Displacement Method
Beams
Part II - Qf, Assembly

Fixed-end Forces / Moments



Indeterminate beam, use superposition!

identify redundant reactions

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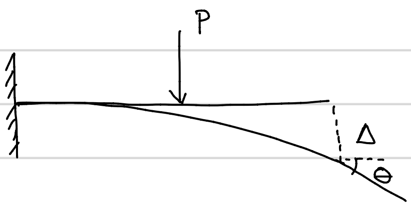

remove redundant reactions and apply loading

$R_1 = R_2 = 0$

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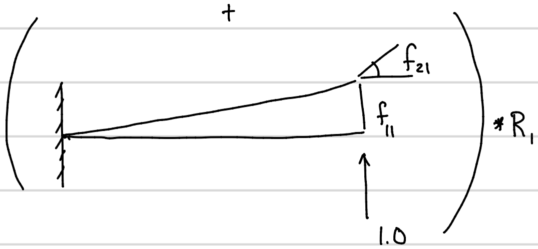

$R_1 = 1.0\ ,\ R_2 = 0$

Note: $\Delta_1 = f_{11} R_1 \qquad \theta_1 = f_{21} R_1$

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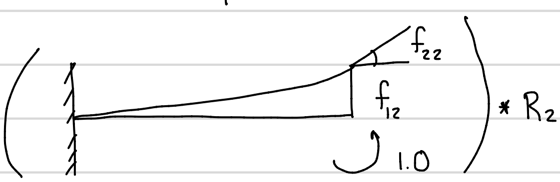
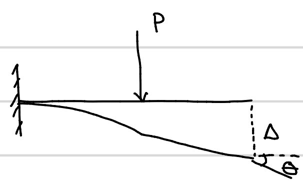


$R_2 = 1.0\ ,\ R_1 = 0$

Note: $\Delta_2 = f_{12} R_2 \qquad \theta_2 = f_{22} R_2$

Compatibility

$$0 = \Delta + \Delta_1 + \Delta_2 \qquad \text{displacement}$$
$$= \Delta + f_{11} R_1 + f_{12} R_2$$



$$\theta = \frac{-P(L/2)^2}{2EI} = \frac{-PL^2}{8EI}$$

$$0 = \theta + \theta_1 + \theta_2 \qquad \text{rotation}$$
$$= \theta + f_{21} R_1 + f_{22} R_2$$

$$\Delta = \frac{-P(L/2)^3}{3EI} + \theta \cdot L/2 = \frac{-5PL^3}{48EI}$$

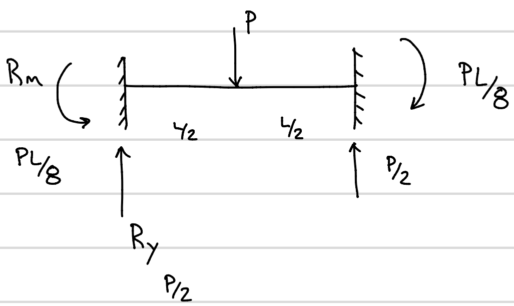
$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{bmatrix} -\Delta \\ -\theta \end{bmatrix}$$

$$\begin{bmatrix} \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{bmatrix} \frac{5PL^3}{48EI} \\ \frac{PL^2}{8EI} \end{bmatrix}$$

Solving system of equations yields

$R_1 = P/2$ fixed-end-force (FEF)

$R_2 = -PL/8$ fixed-end-moment (FEM)

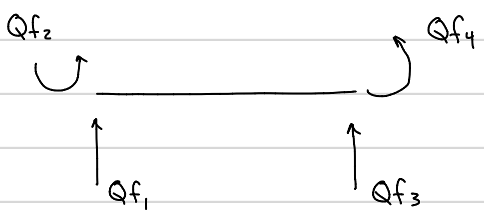


From static equilibrium (determinate)

$\Sigma F_y = 0 \quad R_y + P/2 - P = 0 \quad R_y = P/2$

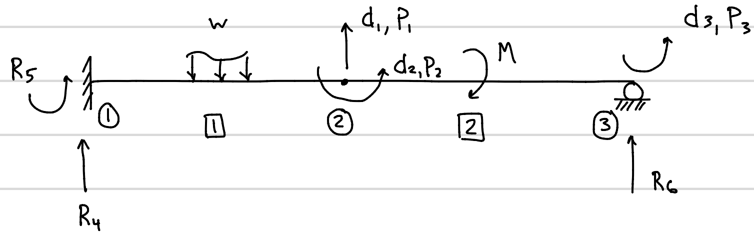
$\Sigma M_b = 0 \quad R_m - P(L/2) + P/2\,(L) - PL/8 = 0$

$R_m = PL/8$



$$\{Q_f\} = \begin{Bmatrix} P/2 \\ PL/8 \\ P/2 \\ -PL/8 \end{Bmatrix} \longleftarrow \text{use correct sign !}$$

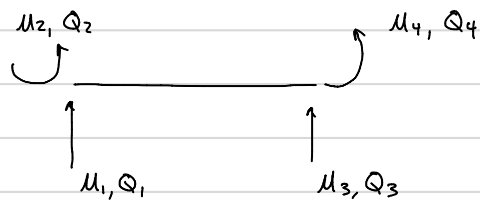
Assembly



1. Rigorous Method (joint equilibrium)

2. Code # Method

code # / member #	u_1, Q_1	u_2, Q_2	u_3, Q_3	u_4, Q_4
1	4	5	1	2
2	1	2	6	3



$$\{Q\} = \{Q_f\} + [k]\{u\}$$

1

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \end{array} \begin{Bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \end{Bmatrix} + \begin{array}{c|cccc} & 4 & 5 & 1 & 2 \\ \hline 4 & k_{11} & k_{12} & k_{13} & k_{14} \\ 5 & k_{21} & k_{22} & k_{23} & k_{24} \\ 1 & k_{31} & k_{32} & k_{33} & k_{34} \\ 2 & k_{41} & k_{42} & k_{43} & k_{44} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

2

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{array}{c} 1 \\ 2 \\ 6 \\ 3 \end{array} \begin{Bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \end{Bmatrix} + \begin{array}{c|cccc} & 1 & 2 & 6 & 3 \\ \hline 1 & k_{11} & k_{12} & k_{13} & k_{14} \\ 2 & k_{21} & k_{22} & k_{23} & k_{24} \\ 6 & k_{31} & k_{32} & k_{33} & k_{34} \\ 3 & k_{41} & k_{42} & k_{43} & k_{44} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\{P\} = \{P_f\} + [S]\{d\}$$

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{Bmatrix} Qf_3^1 + Qf_1^2 \\ Qf_4^1 + Qf_2^2 \\ Qf_4^2 \end{Bmatrix} + \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} k_{33}^1 + k_{11}^2 & k_{34}^1 + k_{12}^2 & k_{14}^2 \\ k_{43}^1 + k_{21}^2 & k_{44}^1 + k_{22}^2 & k_{24}^2 \\ k_{41}^2 & k_{42}^2 & k_{44}^2 \end{bmatrix} \end{array} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

$$\{P - P_f\} = [S]\{d\}$$

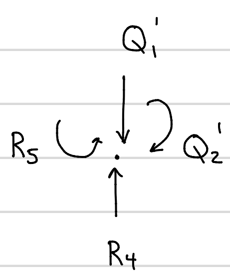
solve : $\{d\} = [S]^{-1}\{P - P_f\}$

post-process :

member-end forces $\{Q\} = \{Q_f\} + [k]\{\underbrace{u}_{\text{compatibility w/ } \{d\}}\}$

reactions joint equilibrium

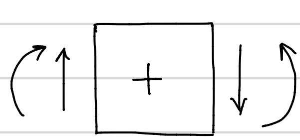
e.g. 1



$\Sigma F_y = 0$ $R_4 = Q_1^1$

$\Sigma M_\circ = 0$ $R_5 = Q_2^1$

shear and moment diagrams



stresses $\sigma_b = -\dfrac{My}{I}$ $\tau = \dfrac{VQ}{Ib}$