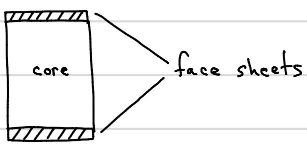
October 03, 2024
Matrix Displacement Method
Beams
Part III - Shear deformation

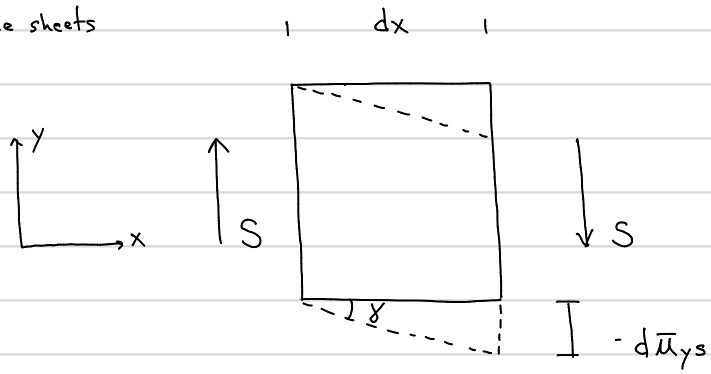
Timoshenko Beam Theory

(includes shear deformation)

Bernoulli: Euler Beam Theory assumes long, slender members where deformations due to shear are negligible compared to bending

However for beams with $L/d < 10$ or built-up members (e.g. sandwich panels) the effects of shear deformation should be included in structural analysis (Timoshenko)





Shear strain (engineering) $\gamma = \dfrac{-d\bar{u}_{ys}}{dx}$

constitutive $\tau = G\gamma$ G - shear modulus   * $G = \dfrac{E}{2(1+\nu)}$

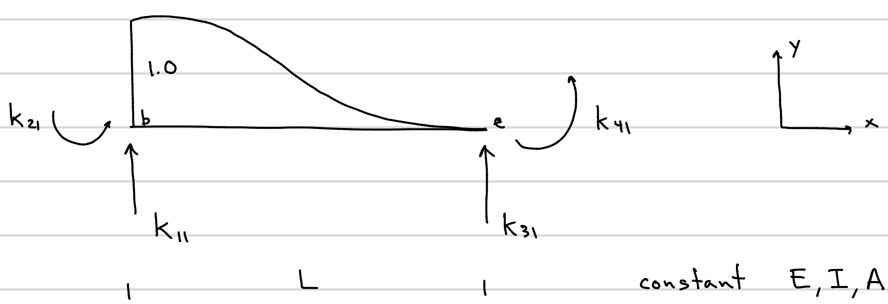
$\tau = f_s \dfrac{S}{A}$ f_s non-dimensional <u>shape factor</u> accounts for non-uniform shear stress distribution on member cross-section

$f_s = 6/5$ for rectangular
$f_s = 10/9$ for circular
$f_s = 1.0$ for an I-beam ($A = A_{web}$)

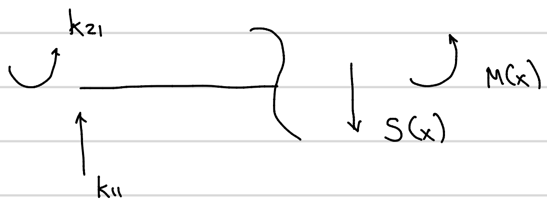
$$\frac{d\bar{u}_{ys}}{dx} = -\frac{\gamma}{G} = -\frac{f_s S}{GA}$$ $\boxed{\text{Shear}}$

$$\frac{d^2\bar{u}_{yB}}{dx^2} = \frac{M}{EI}$$ moment-curvature $\boxed{\text{Bending}}$

$u_1 = 1.0 \quad u_2 = u_3 = u_4 = 0$ (1st column of [k])



constant E, I, A, G



$\sum F_y = 0 \qquad S(x) = k_{11}$

$\sum M = 0 \qquad M(x) = -k_{21} + k_{11}x$

$$\frac{d\bar{u}_{ys}}{dx} = \frac{-f_s S}{GA} = \frac{-f_s k_{11}}{GA}$$

integrating w.r.t. x $\quad * \; \bar{u}_{ys} = -\frac{f_s k_{11} x}{GA} + C_1 \; *$

$$\frac{d^2 \bar{u}_{yB}}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left(-k_{21} + k_{11} x\right)$$

integrating w.r.t. x

$$* \quad \frac{d\bar{u}_{yB}}{dx} = \frac{1}{EI}\left(-k_{21}x + \frac{k_{11}x^2}{2}\right) + C_2 \quad \bullet$$

integrating w.r.t x

$$* \quad \bar{u}_{yB} = \frac{1}{EI}\left(-k_{21}\frac{x^2}{2} + \frac{k_{11}x^3}{6}\right) + C_2 x + C_3$$

$$\bar{u}_y = \underset{\text{shear}}{\bar{u}_{ys}} + \underset{\text{bending}}{\bar{u}_{yB}} = \frac{-f_s k_{11} x}{GA} + \frac{1}{EI}\left(-k_{21}\frac{x^2}{2} + \frac{k_{11}x^3}{6}\right) + C_2 x + \underbrace{C_1 + C_3}_{C_4} \quad \bullet$$

4 unknowns : k_{11}, k_{21}, C_2, C_4

B.C.s

1. $x = 0$, $\theta = 0$ (u_2) • $\Delta = \bar{u}_y$
2. $x = 0$, $\Delta = 1$ (u_1)
3. $x = L$, $\theta = 0$ (u_4) • $\theta = \frac{d\bar{u}_{yB}}{dx}$ (rotation only due to bending)
4. $x = L$, $\Delta = 0$ (u_3)

B.C. 1 $\quad \theta(0) = \frac{d\bar{u}_{yB}}{dx}(0) = 0 \quad \therefore \quad C_2 = 0$

B.C. 2 $\quad \Delta(0) = \bar{u}_y(0) = 1 \quad \therefore \quad C_4 = 1.0$

B.C. 3 $\quad \theta(L) = 0 \quad \frac{1}{EI}\left(-k_{21}L + \frac{k_{11}L^2}{2}\right) + 0 = 0 \qquad k_{21} = \frac{k_{11}L}{2}$

substitute

B.C. 4 $\quad \Delta(L) = 0 \quad \frac{-f_s k_{11} L}{GA} + \frac{1}{EI}\left(\frac{-k_{21}L^2}{2} + \frac{k_{11}L^3}{6}\right) + 1 = 0$

$$k_{11} = \frac{12\,EI}{L^3}\left(\frac{1}{1+B_s}\right) \qquad B_s = \frac{12\,EI\,f_s}{GAL^2}$$

non-dimensional shear deformation constant

$$k_{21} = \frac{6EI}{L^2}\left(\frac{1}{1+B_s}\right)$$

From equilibrium (beam element) $\quad \Sigma F_y = 0 \quad k_{11} + k_{31} = 0 \quad \therefore \quad k_{31} = -k_{11} = \frac{-12EI}{L^3}\left(\frac{1}{1+B_s}\right)$

$$\Sigma M_b = 0 \quad k_{21} + k_{41} + k_{31}L = 0 \quad \therefore \quad k_{41} = \frac{6EI}{L^2}\left(\frac{1}{1+B_s}\right)$$

This completes column 1, remaining columns can be derived in a similar manner

The resulting modified stiffness matrix $[k]$ is :

$$[k] = \frac{EI}{L^3(1+B_s)}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & L^2(4+B_s) & -6L & L^2(2-B_s) \\ -12 & -6L & 12 & -6L \\ 6L & L^2(2-B_s) & -6L & L^2(4+B_s) \end{bmatrix}$$

where $B_s = \frac{12\,EI\,f_s}{GAL^2} \qquad f_s = 1.2$ for rectangle

standard BEBT $[k]$, $B_s = \emptyset$