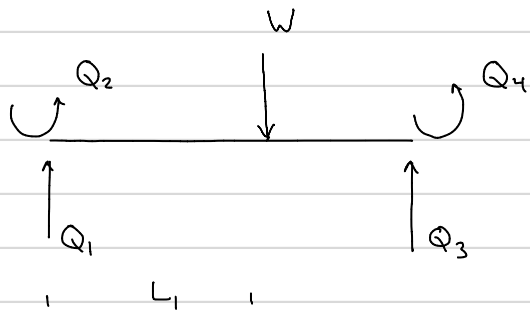
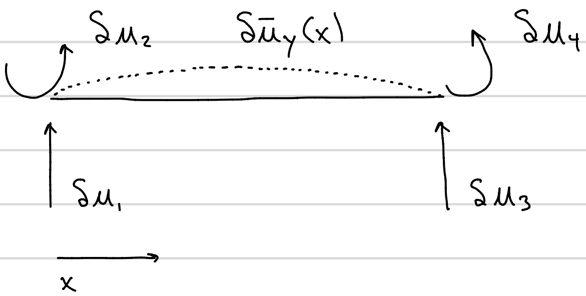
October 08, 2024
Principle of Virtual Work
Beams (BEBT)
Part IV - Finite Element Method

Virtual Work for Beams (BEBT)



$\delta W_{ext} = \delta W_{int}$

real forces * virtual translations
real moments * virtual rotations



$$\delta W_e = Q_1 \delta u_1 + Q_2 \delta u_2 + Q_3 \delta u_3 + Q_4 \delta u_4 - W \delta \bar{u}_y(L_1)$$

$$= \delta u^T Q - W \delta \bar{u}_y(L_1)$$

$$\{\delta u_1 \; \delta u_2 \; \delta u_3 \; \delta u_4\} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

What is $\delta \bar{u}_y(x)$?

Finite Element Method : assume displacement function - complete polynomial

$$\bar{u}_y(x) = \sum_{i=0}^{n} a_i x^i , \; a_i \neq 0 \qquad n = \text{B.C.s} - 1 \qquad \text{beam 4 B.C.s}$$

$\therefore \; n = 3$ cubic

$$\bar{u}_y(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$$

$$\bar{u}_y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

B.C.s $\quad @\ x=0 \quad \bar{u}_y = u_1 \qquad @\ x=L \quad \bar{u}_y = u_3 \qquad$ translational

$\quad @\ x=0 \quad \dfrac{d\bar{u}_y}{dx} = u_2 \qquad @\ x=L \quad \dfrac{d\bar{u}_y}{dx} = u_4 \qquad$ rotational

$$\therefore \quad a_0 = u_1 \qquad a_1 = u_2$$

$$a_2 = \frac{1}{L^2}\left(-3u_1 - 2Lu_2 + 3u_3 - Lu_4\right)$$

$$a_3 = \frac{1}{L^3}\left(2u_1 + Lu_2 - 2u_3 + Lu_4\right)$$

$$\begin{aligned}\bar{u}_y = & \left[1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3\right] u_1 \; + \\ & \left[x\left(1 - \frac{x}{L}\right)^2\right] u_2 \; + \\ & \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right] u_3 \; + \\ & \left[\frac{x^2}{L}\left(\frac{x}{L} - 1\right)\right] u_4\end{aligned}$$

$$\bar{u}_y = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 = \{N\}\{u\}$$

$$\{N_1 \; N_2 \; N_3 \; N_4\} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

* similarly $\delta \bar{u}_y = N\, \delta u$

Bubnov-Galerkin

$$\delta W_{int} = \int_V \delta \varepsilon^T \sigma \, dV \qquad \text{(strain energy)}$$

$$\varepsilon = -y K \qquad {}^{*} K = \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{v''}{[1 + (v')^2]^{3/2}} \approx \frac{d\theta}{dx} = v''$$

$$\varepsilon = -y \frac{d^2 \bar{u}_y}{dx^2} \qquad \bar{u}_y = N u$$

$$\varepsilon = -y \frac{d^2 (N u)}{dx^2} = -y \frac{d^2 N}{dx^2} u = B u$$

$$N = \{ N_1 \quad N_2 \quad N_3 \quad N_4 \}$$

$$B = -y \left\{ \frac{d^2 N_1}{dx^2} \quad \frac{d^2 N_2}{dx^2} \quad \frac{d^2 N_3}{dx^2} \quad \frac{d^2 N_4}{dx^2} \right\}$$

$$B = \frac{-y}{L^2} \left\{ 6\left(\frac{2x}{L} - 1\right) \quad 2L\left(\frac{3x}{L} - 2\right) \quad 6\left(1 - \frac{2x}{L}\right) \quad 2L\left(\frac{3x}{L} - 1\right) \right\}$$

$$\sigma = E\varepsilon = E B u \qquad \text{constitutive relation} \qquad \text{(normal stress-strain)}$$

$$\delta W_{int} = \int_V \delta \varepsilon^T \sigma \, dV = \int \delta \varepsilon^T E B u \, dV$$

$$\downarrow (B \delta u)^T = \delta u^T B^T$$

$$\delta W_{int} = \int_V \delta u^T B^T E B u \, dV = \delta u^T \left(\int_V B^T E B \, dV \right) u$$

Recall MDM $\quad Q = Q_f + k u$

$\quad {}^{*} \; Q - Q_f - k u = 0$

Virtual Work $\quad \delta W_{ext} = \delta W_{int}$

$\quad {}^{*} \; \delta W_{ext} - \delta W_{int} = 0$

$$\delta W_{ext} = \delta u^T Q - W \delta \bar{u}_y (L_1)^*$$

$$\delta W_{int} = \delta u^T \left(\int_V B^T E B \, dV \right) u$$

$$^* \; -W \delta \bar{u}_y (L_1) = -W N(L_1) \delta u$$

$$= -W \delta u^T N^T (L_1)$$

$$= -\delta u^T W N^T (L_1)$$

$$\{N_1 \;\; N_2 \;\; N_3 \;\; N_4\} \begin{Bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \\ \delta u_4 \end{Bmatrix}$$

$$\delta W_{ext} - \delta W_{int} = 0$$

$$\delta u^T Q - \delta u^T W N^T (L_1) - \delta u^T \left(\int_V B^T E B \, dV \right) u$$

$$\delta u^T \left(Q - W N^T (L_1) - \int_V B^T E B \, dV \, u \right) = 0$$

δu^T arbitrarily chosen and not 0 $\therefore$

$$\underbrace{Q - W N^T (L_1)}_{Q_f} - \underbrace{\int_V B^T E B \, dV}_{[k]} u = 0$$

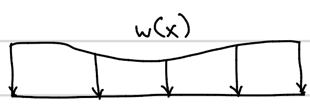
$$[k] = \int_V B^T E B \, dV \qquad B = -y \frac{d^2 N}{dx^2}$$

$$\therefore \quad k = \int_V -y \frac{d^2 N}{dx^2}^T E \; -y \frac{d^2 N}{dx^2} \underbrace{dA \, dx}_{dV}$$

$$k = E \int_V \underbrace{y^2 \, dA} \frac{d^2 N}{dx^2}^T \frac{d^2 N}{dx^2} \, dx$$

$\int y^2 \, dA = I$ second moment of area (moment of inertia)

$$k = EI \int_0^L \frac{d^2 N^T}{dx^2} \frac{d^2 N}{dx^2} \, dx$$



$$W_{ext} = -\int w(x) \, \bar{u}_y(x) \, dx \qquad \bar{u}_y = N u$$

$$\delta W_{ext} = -\int w(x) \, \delta\bar{u}_y(x) \, dx \qquad \delta\bar{u}_y = N \delta u = \delta u^T N^T$$

$$= -\delta u^T \underbrace{\int_0^L w(x) N^T \, dx}_{Q_f}$$