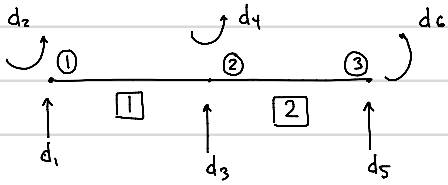
October 09, 2024
Matrix Displacement Method
Beams
Part V - Support displacements

Support Displacements (beam)



	μ_1	μ_2	μ_3	μ_4
$\boxed{1}$	1	2	3	4
$\boxed{2}$	3	4	5	6

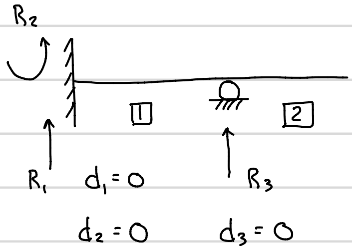
connected coordinates

$$\{P\} = \{P_f\} + [S]\{d\}$$

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} \qquad \{P_f\} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{Bmatrix} Qf_1^1 \\ Qf_2^1 \\ Qf_3^1 + Qf_1^2 \\ Qf_4^1 + Qf_2^2 \\ Qf_3^2 \\ Qf_4^2 \end{Bmatrix} \qquad \{Qf\}^1 = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{Bmatrix} Qf_1^1 \\ Qf_2^1 \\ Qf_3^1 \\ Qf_4^1 \end{Bmatrix} \qquad \{Qf\}^2 = \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{Bmatrix} Qf_1^2 \\ Qf_2^2 \\ Qf_3^2 \\ Qf_4^2 \end{Bmatrix}$$

$$[k]^1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \end{array} \qquad [k]^2 = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \end{array}$$

$$[S] = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & k_{14}^1 & 0 & 0 \\ k_{21}^1 & k_{22}^1 & k_{23}^1 & k_{24}^1 & 0 & 0 \\ k_{31}^1 & k_{32}^1 & k_{33}^1 + k_{11}^2 & k_{34}^1 + k_{12}^2 & k_{13}^2 & k_{14}^2 \\ k_{41}^1 & k_{42}^1 & k_{43}^1 + k_{21}^2 & k_{44}^1 + k_{22}^2 & k_{23}^2 & k_{24}^2 \\ 0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2 & k_{34}^2 \\ 0 & 0 & k_{41}^2 & k_{42}^2 & k_{43}^2 & k_{44}^2 \end{bmatrix}$$



support displacements = 0

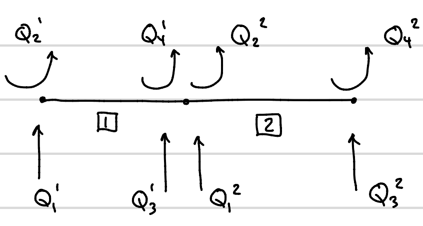
$$
\begin{array}{ll}
\text{unknown} & R_1 \\
\text{loads} & R_2 \\
 & R_3
\end{array}
\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{Bmatrix} Pf_1 \\ Pf_2 \\ Pf_3 \\ Pf_4 \\ Pf_5 \\ Pf_6 \end{Bmatrix} +
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\
S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\
S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\
S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\
S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\
S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66}
\end{bmatrix}
\end{array}
\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix}
\begin{array}{l} = 0 \\ = 0 \\ = 0 \\ \\ \text{unknown} \\ \text{DOFs} \end{array}
$$

$$
\begin{Bmatrix} P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{Bmatrix} Pf_4 \\ Pf_5 \\ Pf_6 \end{Bmatrix} + \begin{bmatrix} S_{44} & S_{45} & S_{46} \\ S_{54} & S_{55} & S_{56} \\ S_{64} & S_{65} & S_{66} \end{bmatrix} \underline{\underline{\begin{Bmatrix} d_4 \\ d_5 \\ d_6 \end{Bmatrix}}}
$$

solve for DOFs

calculate reactions
$$
\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{Bmatrix} Pf_1 \\ Pf_2 \\ Pf_3 \end{Bmatrix} + \begin{bmatrix} S_{14} & S_{15} & S_{16} \\ S_{24} & S_{25} & S_{26} \\ S_{34} & S_{35} & S_{36} \end{bmatrix} \begin{Bmatrix} d_4 \\ d_5 \\ d_6 \end{Bmatrix}
$$

then substitute



$$
\begin{Bmatrix} Qf_1^1 \\ Qf_2^1 \\ Qf_3^1 + Qf_1^2 \end{Bmatrix} \qquad \begin{bmatrix} k_{14}^1 & 0 & 0 \\ k_{24}^1 & 0 & 0 \\ k_{34}^1 + k_{12}^2 & k_{13}^2 & k_{14}^2 \end{bmatrix}
$$

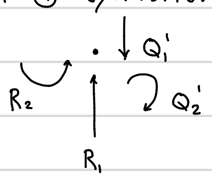
* $R_1 = Qf_1^1 + k_{14}^1 d_4$

$R_2 = Qf_2^1 + k_{24}^1 d_4$

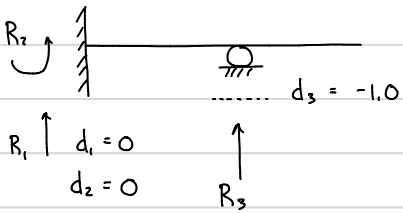
$R_3 = \ldots\ldots$

same result

joint ① equilibrium



$$
\begin{array}{r} R_1 = Q_1^1 \longrightarrow \\ R_2 = Q_2^1 \longrightarrow \\ \\ \\ \end{array}
\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} Qf_1 \\ Qf_2 \\ Qf_3 \\ Qf_4 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \begin{array}{l} = 0 \\ = 0 \\ = 0 \\ = d_4 \end{array}
$$



support displacements $\neq 0$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \\ \\ \\ \\ \end{matrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{Bmatrix} Pf_1 \\ Pf_2 \\ Pf_3 \\ Pf_4 \\ Pf_5 \\ Pf_6 \end{Bmatrix} + \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} \begin{matrix} = 0 \\ = 0 \\ = -1.0 \\ \\ \text{unknown} \\ \\ \end{matrix}$$

$$\underbrace{\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix}}_{\text{unknown}} = \begin{Bmatrix} Pf_1 \\ Pf_2 \\ Pf_3 \end{Bmatrix} + \begin{Bmatrix} S_{13} \\ S_{23} \\ S_{33} \end{Bmatrix} \underbrace{d_3}_{\text{known}} + \begin{bmatrix} S_{14} & S_{15} & S_{16} \\ S_{24} & S_{25} & S_{26} \\ S_{34} & S_{35} & S_{36} \end{bmatrix} \underbrace{\begin{Bmatrix} d_4 \\ d_5 \\ d_6 \end{Bmatrix}}_{\text{unknown}}$$

3 equations
6 unknowns

$$\underbrace{\begin{Bmatrix} P_4 \\ P_5 \\ P_6 \end{Bmatrix}}_{\text{known}} = \underbrace{\begin{Bmatrix} Pf_4 \\ Pf_5 \\ Pf_6 \end{Bmatrix}}_{\downarrow \{Pf\}} + \underbrace{\begin{Bmatrix} S_{43} \\ S_{53} \\ S_{63} \end{Bmatrix} \underbrace{d_3}_{\text{known}}}_{\downarrow \{Pf_s\}} + \begin{bmatrix} S_{44} & S_{45} & S_{46} \\ S_{54} & S_{55} & S_{56} \\ S_{64} & S_{65} & S_{66} \end{bmatrix} \underbrace{\begin{Bmatrix} d_4 \\ d_5 \\ d_6 \end{Bmatrix}}_{\text{unknown}}$$

3 equations
3 unknowns

$\{Pf_s\}$: fixed-end forces due to support displacements

$$\{Qf_s\}^* \equiv [k]\{u f_s\}^*$$

$$\{Pf_s\} = \begin{Bmatrix} S_{43} \\ S_{53} \\ S_{63} \end{Bmatrix} d_3 = \begin{Bmatrix} k_{43}^1 + k_{21}^2 \\ k_{31}^2 \\ k_{41}^2 \end{Bmatrix} d_3$$

$$\{u f_s\}^1 = \begin{Bmatrix} 0 \\ 0 \\ d_3 \\ 0 \end{Bmatrix} \begin{matrix} \\ \\ = -1.0 \\ \\ \end{matrix} \qquad \{u f_s\}^2 = \begin{Bmatrix} d_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \begin{matrix} = -1.0 \\ \\ \\ \\ \end{matrix}$$

$$^{*}\ \{Q_{fs}\} = [k]\{u_{fs}\}\ ^{*}$$

$$\{Q_{fs}\}^1 = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ d_3 \\ 0 \end{Bmatrix} \qquad \{Q_{fs}\}^1 = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{Bmatrix} Q_{fs1}^1 \\ Q_{fs2}^1 \\ Q_{fs3}^1 \\ Q_{fs4}^1 \end{Bmatrix}$$

$$\{Q_{fs}\}^2 = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} d_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \{Q_{fs}\}^2 = \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{Bmatrix} Q_{fs1}^2 \\ Q_{fs2}^2 \\ Q_{fs3}^2 \\ Q_{fs4}^2 \end{Bmatrix}$$

$$\underline{\underline{\{P_{fs}\}}} = \begin{Bmatrix} P_{fs4} \\ P_{fs5} \\ P_{fs6} \end{Bmatrix} = \begin{array}{c} 4 \\ 5 \\ 6 \end{array} \begin{Bmatrix} Q_{fs4}^1 + Q_{fs2}^2 \\ Q_{fs3}^2 \\ Q_{fs4}^2 \end{Bmatrix} = \begin{Bmatrix} k_{43}^1 + k_{21}^2 \\ k_{31}^2 \\ k_{41}^2 \end{Bmatrix} d_3$$

assembled from member contributions
(identical to prior result)

In general, two approaches to account for support displacements:

<u>1st</u> Establish MDM equations for DOFs only

$$\{P\} = \{P_f\} + [S]\{d\}$$

includes
a) fixed-end forces due to interior member loads
b) support displacements $\{Q_{fs}\} = [k]\{u_{fs}\}$ (local)
$\{F_{fs}\} = [K]\{v_{fs}\}$ (global)

(assemble via code #)

Solve $\{P - P_f\} = [S]\{d\}$
Calculate reactions $\{R\}$ from joint equilibrium

2nd Establish MDM equations for all coordinates

$$\{P\}^* = \{P_f\} + [S]\{d\}^*$$

$$\left\{\begin{array}{c} P \\ \hdashline R \end{array}\right\} = \left\{\begin{array}{c} P_f \\ \hdashline R_f \end{array}\right\} + \left[\begin{array}{c:c} S_{FF} & S_{FR} \\ \hdashline S_{RF} & S_{RR} \end{array}\right] \left\{\begin{array}{c} d_F \\ \hdashline d_R \end{array}\right\}$$ — DOFs

F - free

R - restrained

$\{P_f\}$, $\{R_f\}$: only due to member loading

$\{d_R\}$: support displacements

$$\{P\} = \{P_f\} + [S_{FF}]\underbrace{\{d_F\}}_{\text{unknown}} + [S_{FR}]\{d_R\}$$

$$\underbrace{\{R\}}_{\text{unknown}} = \{R_f\} + [S_{RF}]\underbrace{\{d_F\}}_{\text{unknown}} + [S_{RR}]\{d_R\}$$

Staggered Solution

1st solve for $\underline{\{d_F\}} = [S_{FF}]^{-1}\{P - P_f - [S_{FR}]\{d_R\}\}$

2nd calculate $\underline{\{R\}} = \{R_f\} + [S_{RF}]\{d_F\} + [S_{RR}]\{d_R\}$