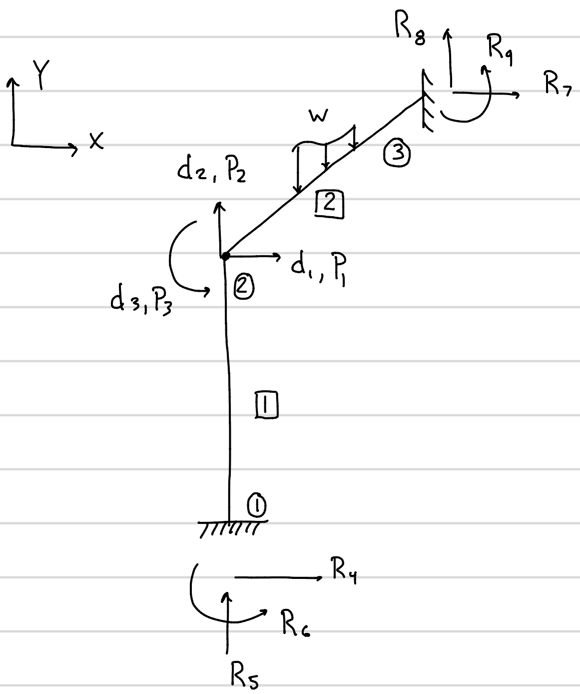
October 17, 2024
Matrix Displacement Method
2D Frames
Part I - Formulation

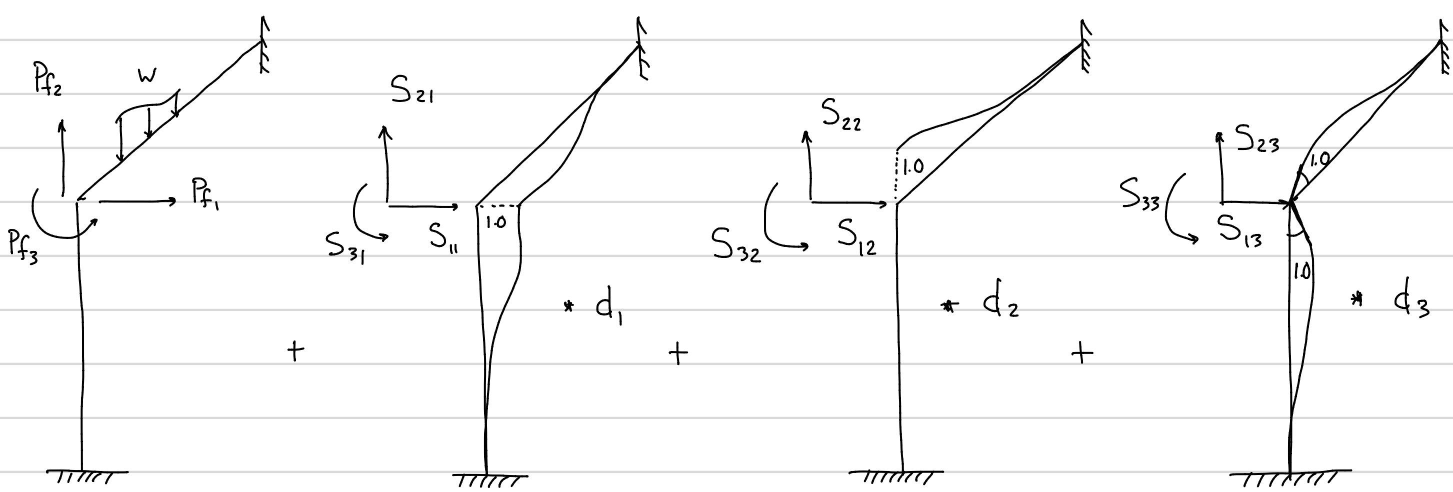
2D Frame Analysis



DOFs : 2 translations X, Y
1 rotation about Z

Structural-Level Superposition

=



Apply member loads $d_1 = d_2 = d_3 = 0$

$d_1 = 1.0, \ d_2 = d_3 = 0$

$d_2 = 1.0, d_1 = d_3 = 0$

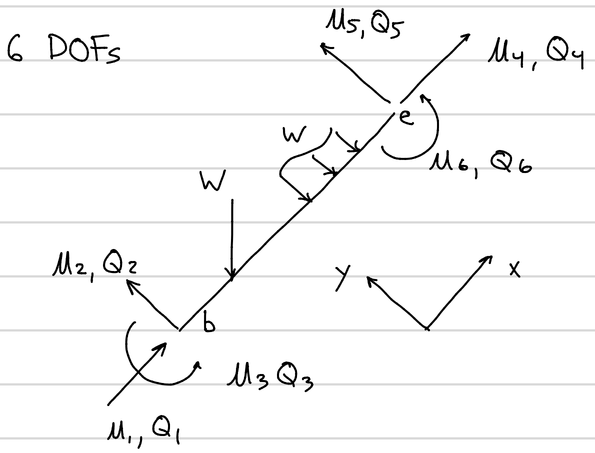
$d_3 = 1.0, \ d_1 = d_2 = 0$

From joint equilibrium

$$\{P\} = \{P_f\} + [S]\{d\}$$

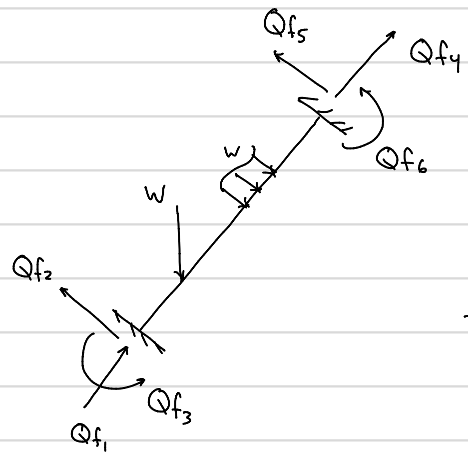
Member-Level Superposition

6 DOFs



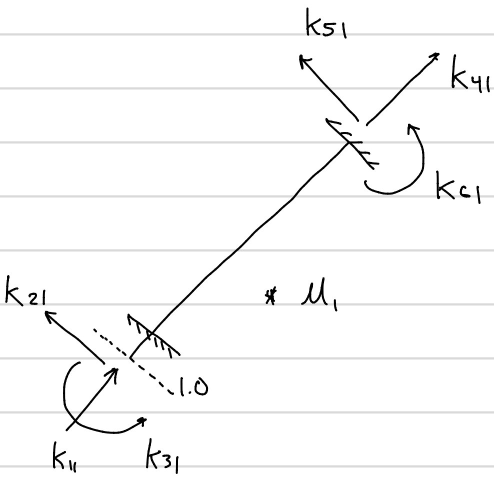
local coordinates
x, y

=



Apply member loads
$u_1 \ldots u_6 = 0$

+



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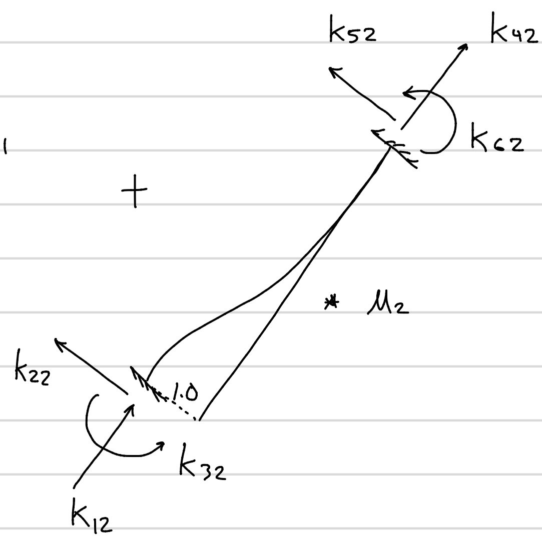

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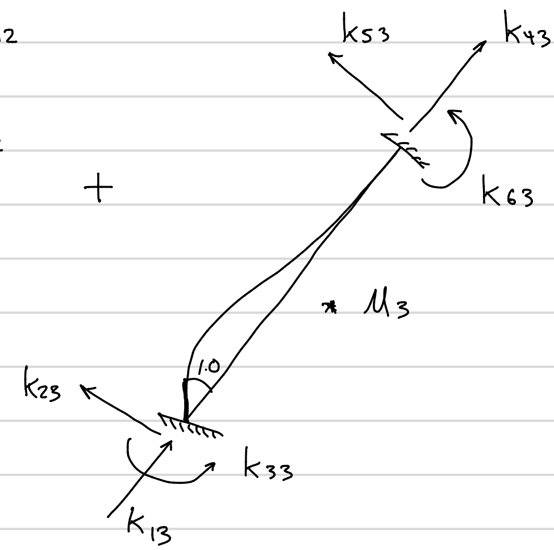


$u_1 = 1,\ u_2 \ldots u_6 = 0$ $\qquad$ $u_2 = 1.0,\ u_1, u_3 \ldots u_6 = 0$ $\qquad$ $u_3 = 1.0,\ u_1, u_2, u_4 \ldots u_6 = 0$

Similarly, $u_4, u_5, u_6 = 1.0$ can be constructed

From member-level superposition

$$\{Q\} = \{Q_f\} + [k]\{u\} \qquad (\text{local})$$

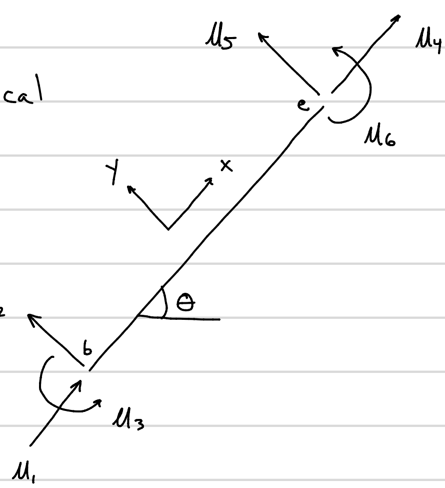
$$\underset{6\times1}{} \quad \underset{6\times1}{} \quad \underset{6\times6}{} \quad \underset{6\times1}{}$$

6x1 6x1 6x6 6x1

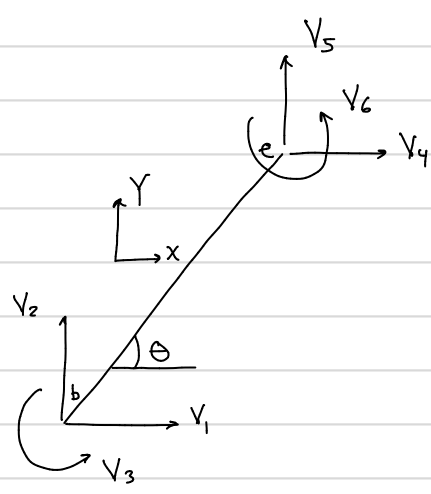
Using same analysis procedures to derive uniaxial/beam stiffness terms, [k] for plane frame :

$$[k] = \begin{bmatrix} EA/L & 0 & 0 & -EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -EA/L & 0 & 0 & EA/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix}$$

local



global



$$u_1 = v_1 \cos\theta + v_2 \sin\theta$$
$$u_2 = -v_1 \sin\theta + v_2 \cos\theta$$
$$u_3 = v_3$$

$$\{u\} = [T]\{v\}$$

$$c = \cos\theta$$
$$s = \sin\theta$$

$$\underset{6\times6}{[T]} = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\{u\} = [T]\{v\}$$

$$\{Q\} = [T]\{F\} \qquad \{F\} = [T]^T\{Q\}$$

$$= \{Q_f\} + [k]\{u\}$$

$$[T]\{v\}$$

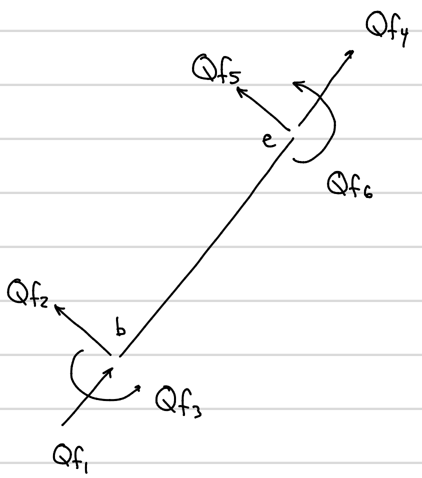
$$\{F\} = \underbrace{[T]^T\{Q_f\}}_{\{F_f\}} + \underbrace{[T]^T[k][T]}_{[K]_{global}}\{v\}$$

Fixed-end force vector in global coordinates

$$\{F\} = \{F_f\} + [K]\{v\} \qquad \text{global, member-level}$$

ASSEMBLY

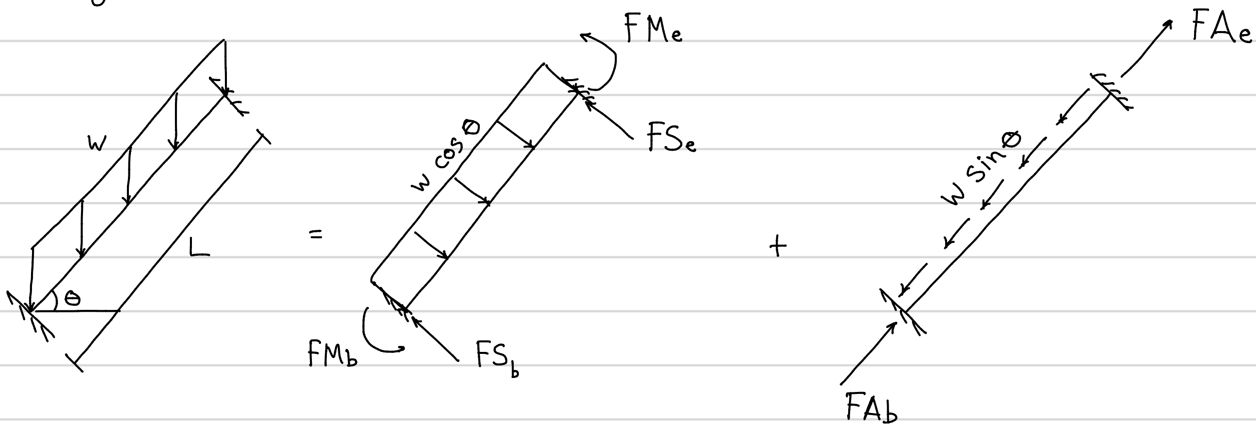
$$\{P\} = \{P_f\} + [S]\{d\} \qquad \text{global, structural-level}$$



$$\{Q_f\} = \begin{Bmatrix} FA_b \\ FS_b \\ FM_b \\ FA_e \\ FS_e \\ FM_e \end{Bmatrix}$$

Kassimali notation

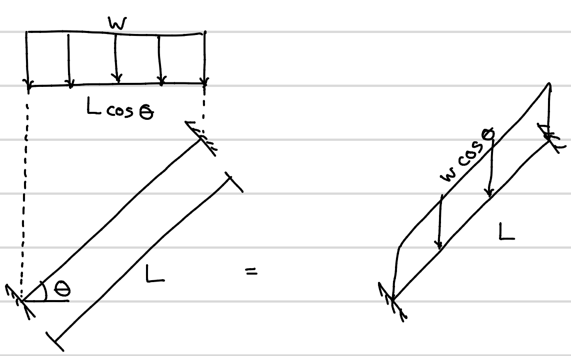
Self-weight loads



$$\{Q_f\} = \begin{Bmatrix} \tfrac{1}{2} wL\sin\theta \\ \tfrac{1}{2} wL\cos\theta \\ \tfrac{1}{12} wL^2\cos\theta \\ \tfrac{1}{2} wL\sin\theta \\ \tfrac{1}{2} wL\cos\theta \\ -\tfrac{1}{12} wL^2\cos\theta \end{Bmatrix} = \frac{wL}{2}\begin{Bmatrix} \sin\theta \\ \cos\theta \\ \tfrac{1}{6} L\cos\theta \\ \sin\theta \\ \cos\theta \\ -\tfrac{1}{6} L\cos\theta \end{Bmatrix}$$

$$\cos\theta = \frac{L_x}{L} \qquad \sin\theta = \frac{L_y}{L}$$

Projected loads (e.g. snow loading)



$$\{Q_f\} = \frac{wL}{2}\cos\theta \begin{Bmatrix} \sin\theta \\ \cos\theta \\ \tfrac{1}{6} L\cos\theta \\ \sin\theta \\ \cos\theta \\ -\tfrac{1}{6} L\cos\theta \end{Bmatrix}$$

* Once $\{Q_f\}$ is determined $\longrightarrow$ $\underset{\text{global}}{\{F_f\}} = [T]^T \underset{\text{local}}{\{Q_f\}}$, $\underset{\text{global}}{[K]} = [T]^T \underset{\text{local}}{[k]} [T]$

then assemble $\{P_f\}$ from $\{F_f\}$, $[S]$ from $[K]$, and solve for $\{d\}$