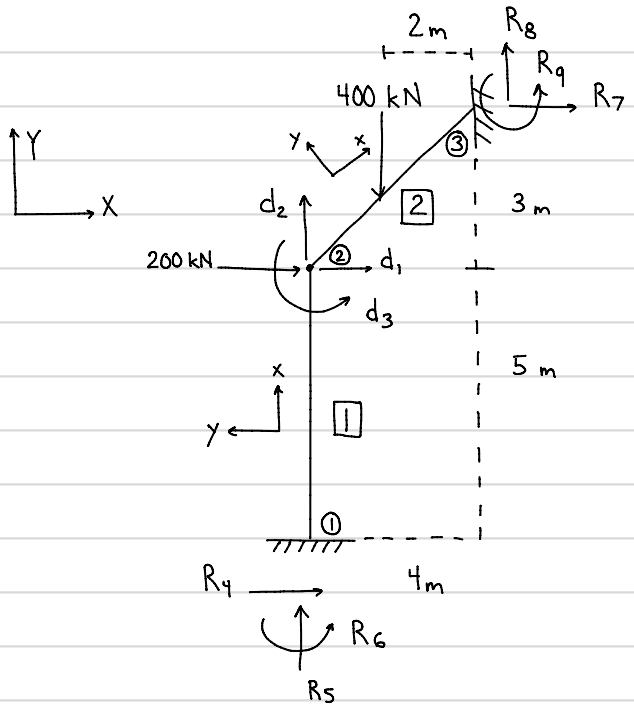
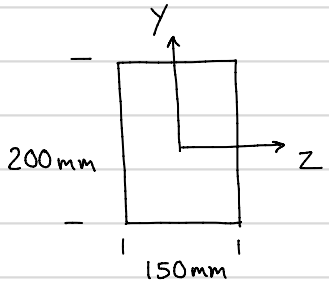
October 16, 2026
Matrix Displacement Method
2D Frames
Part II - Numerical Example

2D Frame Example

1. Label Structure





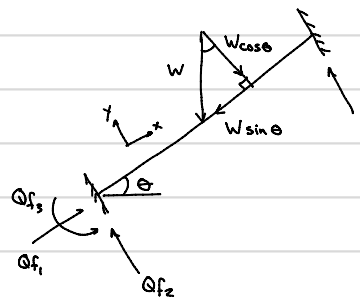
(strong axis bending)

E = 20 GPa for all members

2. $\{P\}_{3\times1} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} 200 \\ 0 \\ 0 \end{Bmatrix}$ kN

3. Member-level contributions (local)

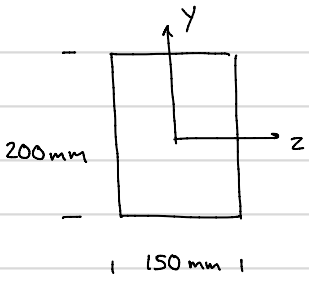
$\{Qf\}^1 = \{\emptyset\}$



$$\{Qf\}^2 = \begin{Bmatrix} FAb \\ FSb \\ FMb \\ FAe \\ FSe \\ FMe \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} W \sin\theta \\ \frac{1}{2} W \cos\theta \\ \frac{1}{8} WL\cos\theta \\ \frac{1}{2} W \sin\theta \\ \frac{1}{2} W \cos\theta \\ -\frac{1}{8} WL\cos\theta \end{Bmatrix} = \begin{Bmatrix} 120 \\ 160 \\ 2e5 \\ 120 \\ 160 \\ -2e5 \end{Bmatrix} \begin{matrix} kN \\ kN \\ kN\cdot mm \\ kN \\ kN \\ kN\cdot mm \end{matrix}$$

$$[k] = \begin{bmatrix} EA/L & 0 & 0 & -EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -EA/L & 0 & 0 & EA/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix}$$

$E = 20\ GPa \quad (GPa = kN/mm^2)$



$A = (150)(200) = 3e4\ mm^2$

$I_z = \frac{(150)(200)^3}{12} = 1e8\ mm^4$

$L^1 = L^2 = 5{,}000\ mm$

$$[k]^{1,2} = \begin{bmatrix} 120 & 0 & 0 & -120 & 0 & 0 \\ 0 & 0.192 & 480 & 0 & -0.192 & 480 \\ 0 & 480 & 1.6e6 & 0 & -480 & 0.8e6 \\ -120 & 0 & 0 & 120 & 0 & 0 \\ 0 & -0.192 & -480 & 0 & 0.192 & -480 \\ 0 & 480 & 0.8e6 & 0 & -480 & 1.6e6 \end{bmatrix}$$

transformation

$$[T] = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

[1] $c = 0$, $s = 1$

[2] $c = 4/5$, $s = 3/5$

$c \equiv \cos\theta = L_x/L$

$s \equiv \sin\theta = L_y/L$

(global) $\{F_f\} = [T]^T \{Q_f\}$

$$\{F_f\}^1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \{F_f\}^2 = \begin{Bmatrix} 0 \\ 200 \\ 2e5 \\ 0 \\ 200 \\ -2e5 \end{Bmatrix} \begin{matrix} kN \\ kN \\ kN \cdot mm \\ kN \\ kN \\ kN \cdot mm \end{matrix}$$

$$[K]_{global} = [T]^T [k] [T]$$

$$[k]^1 = \begin{bmatrix} 0.192 & 0 & -480 & -0.192 & 0 & -480 \\ 0 & 120 & 0 & 0 & -120 & 0 \\ -480 & 0 & 1.6e6 & 480 & 0 & 0.8e6 \\ -0.192 & 0 & 480 & 0.192 & 0 & 480 \\ 0 & -120 & 0 & 0 & 120 & 0 \\ -480 & 0 & 0.8e6 & 480 & 0 & 1.6e6 \end{bmatrix}$$

$$[k]^2 = \begin{bmatrix} 76.87 & 57.51 & -288 & -76.87 & -57.51 & -288 \\ 57.51 & 43.32 & 384 & -57.51 & -43.32 & 384 \\ -288 & 384 & 1.6e6 & 288 & -384 & 0.8e6 \\ -76.87 & -57.51 & 288 & 76.87 & 57.51 & 288 \\ -57.51 & -43.32 & -384 & -57.51 & 43.32 & -384 \\ -288 & 384 & 0.8e6 & 288 & -384 & 1.6e6 \end{bmatrix}$$

4. Assembly (code # method)

$$\{P_f\} = \begin{Bmatrix} F_{f_4}^{1} + F_{f_1}^{2} \\ F_{f_5}^{1} + F_{f_2}^{2} \\ F_{f_6}^{1} + F_{f_3}^{2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 200 \\ 2e5 \end{Bmatrix} \begin{matrix} kN \\ kN \\ kN \cdot mm \end{matrix}$$

$$[S] = \begin{bmatrix} K_{44}^{1} + K_{11}^{2} & K_{45}^{1} + K_{12}^{2} & K_{46}^{1} + K_{13}^{2} \\ K_{54}^{1} + K_{21}^{2} & K_{55}^{1} + K_{22}^{2} & K_{56}^{1} + K_{23}^{2} \\ K_{64}^{1} + K_{31}^{2} & K_{65}^{1} + K_{32}^{2} & K_{66}^{1} + K_{33}^{2} \end{bmatrix} = \begin{bmatrix} 77.06 & 57.51 & 192 \\ 57.51 & 163.32 & 384 \\ 192 & 384 & 3.2e6 \end{bmatrix}$$

5. Solve $\{P\} = \{P_f\} + [S]\{d\}$

$\{d\} = [S]^{-1}\{P - P_f\}$

$$\{d\} = \begin{Bmatrix} 4.8224 \\ -2.7757 \\ -0.06246 \end{Bmatrix} \begin{matrix} mm & d_1 \\ mm & d_2 \\ rad & d_3 \end{matrix}$$

Post-Process

6. Compute member-end forces (global): $\{F\} = \{F_f\} + [k]\{v\} \longrightarrow$ compatibility w/ $\{d\}$

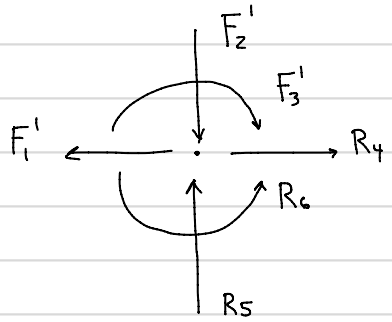
$$\{v\}^1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ d_1 \\ d_2 \\ d_3 \end{Bmatrix} \text{ fixed B.C.} \qquad \{v\}^2 = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \text{ fixed B.C.}$$

$$\{F\}^1 = \begin{Bmatrix} 29.05 \\ 333.09 \\ -47.65 \\ -29.05 \\ -333.09 \\ -97.62 \end{Bmatrix} \begin{matrix} kN \\ kN \\ kN \cdot m^* \\ kN \\ kN \\ kN \cdot m^* \end{matrix} \qquad \{F\}^2 = \begin{Bmatrix} 229.05 \\ 333.09 \\ 97.62 \\ -229.05 \\ 66.91 \\ -252.42 \end{Bmatrix} \begin{matrix} kN \\ kN \\ kN \cdot m^* \\ kN \\ kN \\ kN \cdot m^* \end{matrix}$$

* converted from $kN \cdot mm$

7. Support reactions (joint equilibrium)

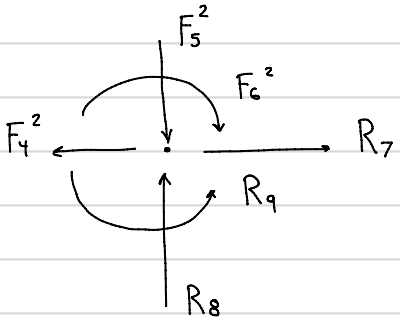
$$\{F\}^1 = \begin{Bmatrix} 29.05 \\ 333.09 \\ -47.65 \\ -29.05 \\ -333.09 \\ -97.62 \end{Bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN·m} \\ \text{kN} \\ \text{kN} \\ \text{kN·m} \end{matrix}^* \qquad \{F\}^2 = \begin{Bmatrix} 229.05 \\ 333.09 \\ 97.62 \\ -229.05 \\ 66.91 \\ -252.42 \end{Bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN·m} \\ \text{kN} \\ \text{kN} \\ \text{kN·m} \end{matrix}^*$$

① 

$\Sigma F_x = 0 \quad R_4 = F_1^1 = 29.05 \text{ kN}$

$\Sigma F_y = 0 \quad R_5 = F_2^1 = 333.09 \text{ kN}$

$\Sigma M_\cdot = 0 \quad R_6 = F_3^1 = -47.65 \text{ kN·m} \quad \therefore \circlearrowright$

③ 

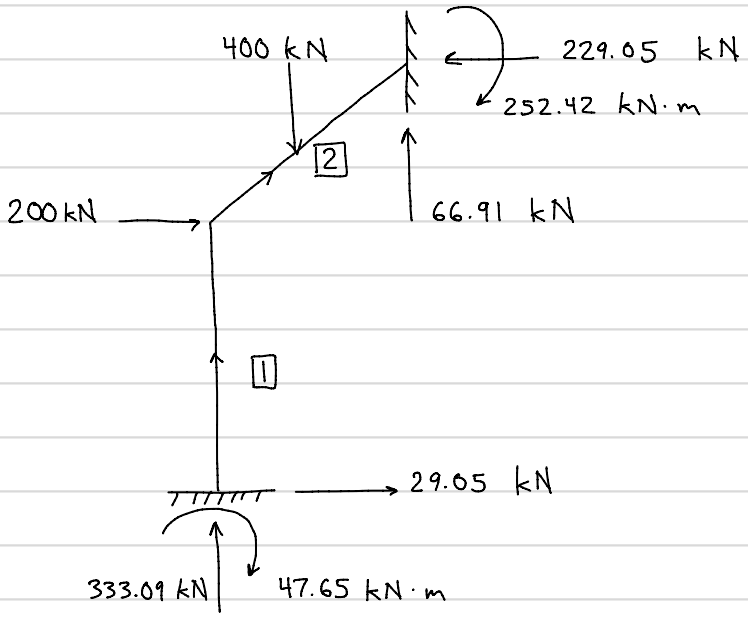
$\Sigma F_x = 0 \quad R_7 = F_4^2 = -229.05 \text{ kN} \quad \therefore \leftarrow$

$\Sigma F_y = 0 \quad R_8 = F_5^2 = 66.91 \text{ kN}$

$\Sigma M_\cdot = 0 \quad R_9 = F_6^2 = -252.42 \text{ kN·m} \quad \therefore \circlearrowright$

* Check overall equilibrium *

8. Draw axial force, shear force, and bending moment diagrams



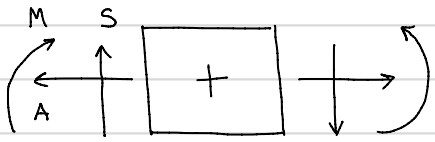
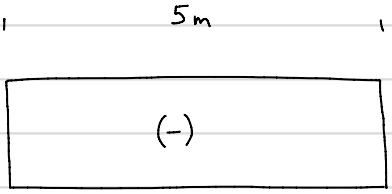


diagram positive sign convention

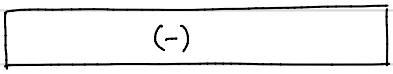
Member 1

Axial



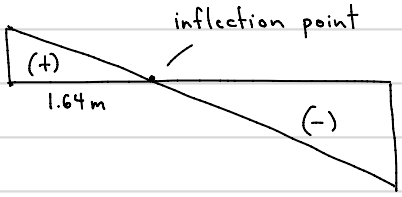
333.09 kN (compression)

Shear



29.05 kN

Moment

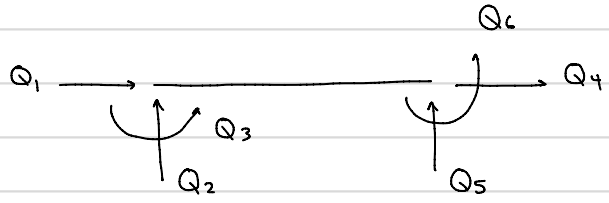


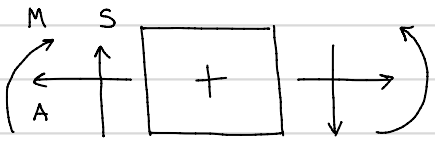
inflection point (i.e., change in curvature / concavity)

97.6 kN·m

Member [2] (easier to work with Q) $\{Q\} = [T]\{F\}$

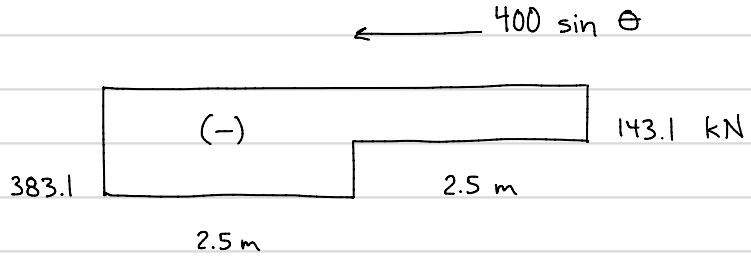
$$\{Q\}^2 = \begin{Bmatrix} 383.1 \\ 129.04 \\ 97.6 \\ -143.1 \\ 191 \\ -252.42 \end{Bmatrix} \begin{matrix} kN \\ kN \\ kN \cdot m \\ kN \\ kN \\ kN \cdot m \end{matrix} \begin{matrix} \text{axial} \\ \text{shear} \\ \text{moment} \end{matrix}$$



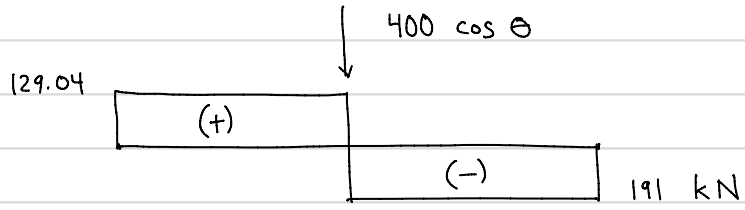


* don't forget interior loads *

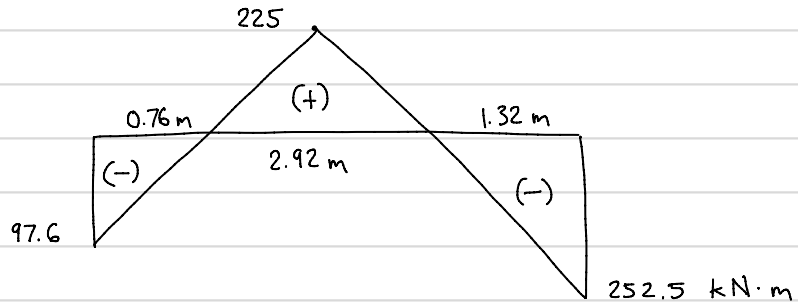
Axial



Shear



Moment

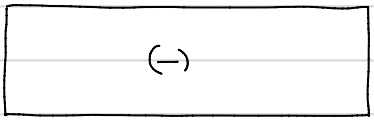


9. Calculate stresses

Member [1]

Axial Stress

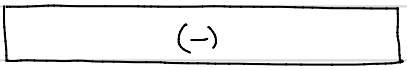
$\sigma_a = {}^{Force}/_{Area}$



11.1 MPa

$Area = 3e4\ mm^2$

Shear Stress

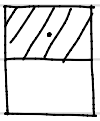


1.45 MPa

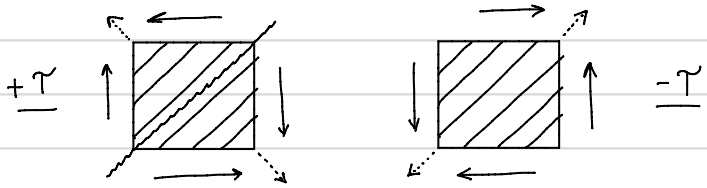
$\tau = \frac{VQ}{Ib}$

* sign of shear stress important when:
- combining w/ torsion effects
- anisotropic materials (e.g. fiber-composites)

$I = 1e8\ mm^4$

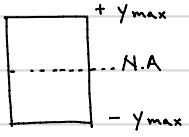


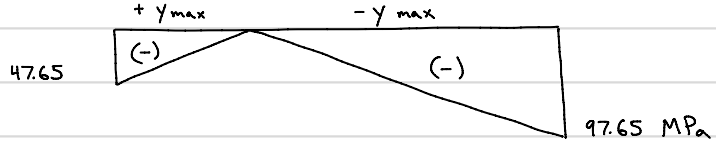
$Q = (150 * 100)(50)$
$= 7.5e5\ mm^3$



Bending Stress

$\sigma_b = \frac{-My}{I}$ * $\sigma = \sigma_a + \sigma_b$ ∴ we need to consider worst case combination(s)





* $\sigma_{max} = -11.1 - 97.65 = -108.75$ MPa (compression) * $\tau_{max} = 1.45$ MPa

σ_{max} (tension) $= -11.1 + 97.65 = 86.55$ MPa

$+y_{max}$

Member [2]

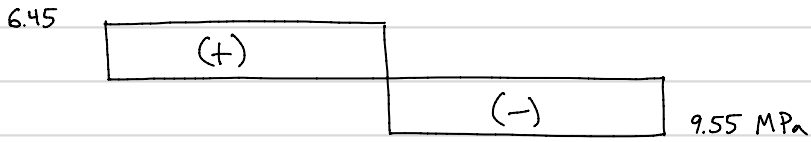
Axial Stress

$$\sigma_a = \frac{P}{A}$$



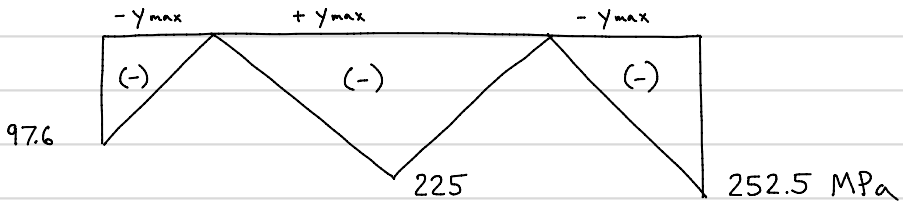
Shear Stress

$$\tau = \frac{VQ}{Ib}$$



Bending Stress

$$\sigma_b = -\frac{My}{I}$$



σ_{max} = 257.27 MPa (compression) τ_{max} = 9.55 MPa