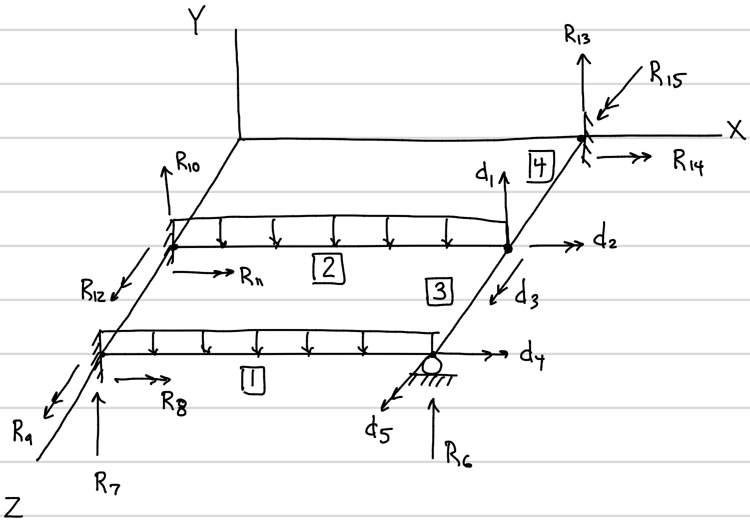
October 22, 2024
Matrix Displacement Method
Grids
Part I - Formulation

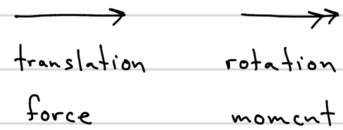
Grids

Two-dimensional framework subjected to loads and reactions perpendicular to the plane of the structure (e.g. supporting structures for roofs/floors)

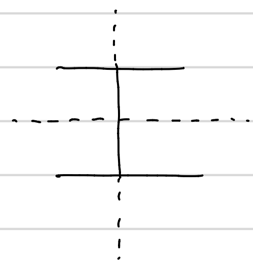
Formulation of grids is in horizontal plane
(plane frames in vertical plane)

DOFs 1 translation
2 rotations

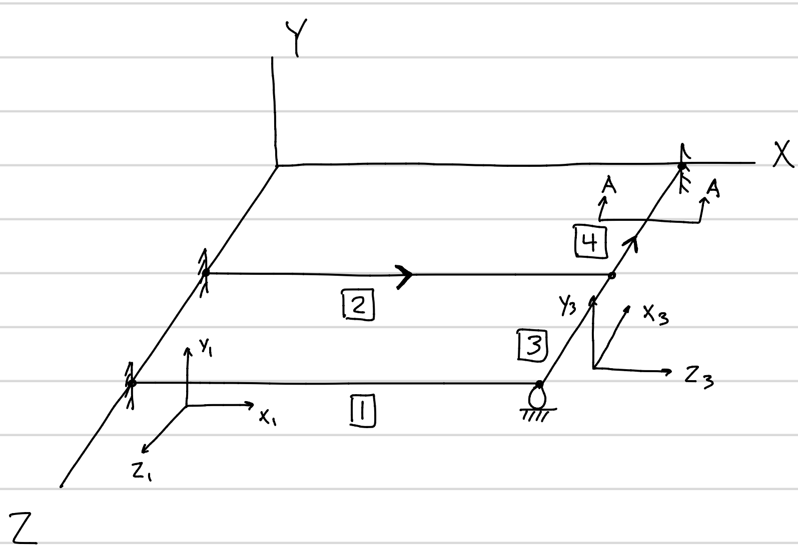


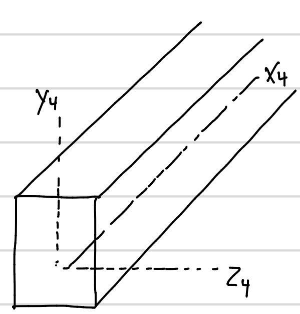


Grid members have doubly symmetric cross-sections



$$\{P\} = \{P_f\} + [S]\{d\}$$ structural-level system of linear equations



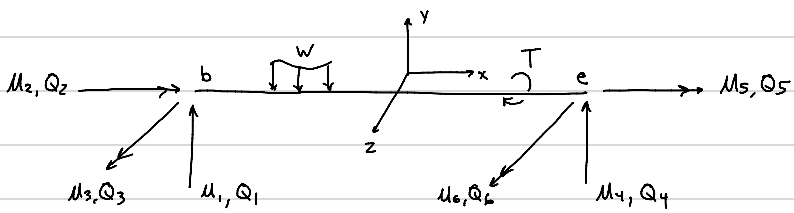


Section A-A

global Y and local y axes coincide!

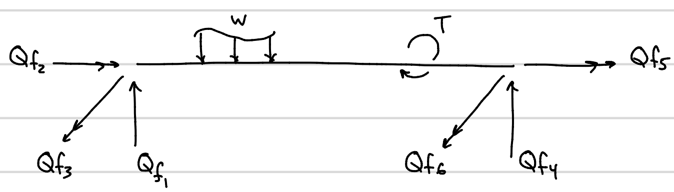
When a non-circular member is subjected to torsion, initially plane cross-sections become warped surfaces. Restraint of warping (out-of-plane deformation) can induce bending stresses in member.

Assumption in analysis of grids (and space frames next) is that cross-sections are free to warp out of plane under action of torsional moments ∴ bending and torsion are uncoupled



$$\underset{6\times1}{\{Q\}} = \{Q_f\} + \underset{6\times6}{[k]}\{u\}$$

=



Apply loads
$(u_1 \ldots u_6 = 0)$

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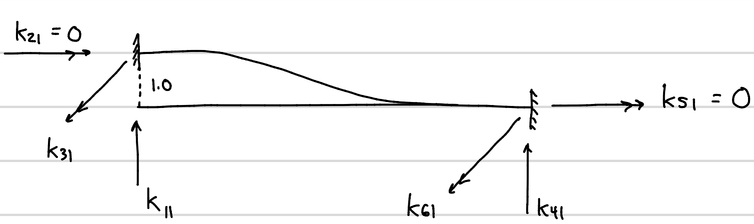
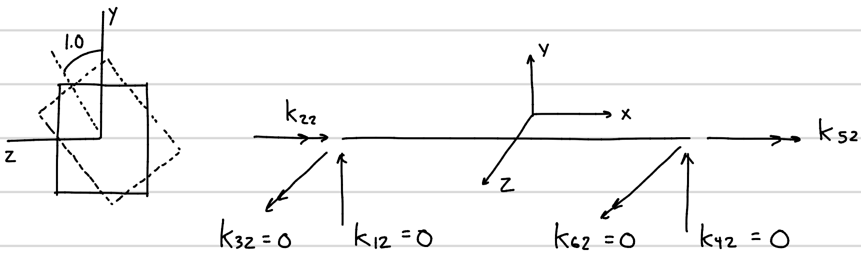


$u_1 = 1.0, \; u_2 \ldots u_6 = 0$

Remaining terms from 1st column of [k] beam



$u_2 = 1.0,\ u_1, u_3 \ldots u_6 = 0$

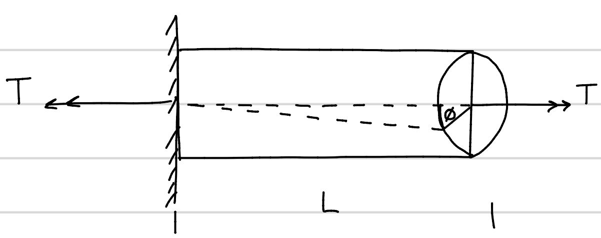
bending and torsion are <u>uncoupled</u>

+

$u_3 = 1.0,\ u_1, u_2, u_4 \ldots u_6 = 0$

Remaining terms from 2nd column of [k] beam

Determine k_{22} and k_{52} (torsional stiffness constants)



ø angle of twist

$\phi = \dfrac{TL}{GJ}$ J = polar moment of inertia $\int \rho^2 \, dA$ $J_{circle} = \int_0^{2\pi} \int_0^R \rho^2 \, r \, dr \, d\theta$ ($\rho^2 \to r^2$)

$J_{circle} = \dfrac{\pi R^4}{2}$ $J_{cylinder} = 2\pi R^3 t$

for unrestrained warping in non-circular cross-sections

J_{polar} = Saint-Venant's torsion constant, $J_{rectangle} = \beta b^3 d$ for $b \le d$ $\beta = \frac{1}{3} - 0.21 \frac{b}{d}\left[1 - \frac{1}{12}\left(\frac{b}{d}\right)^4\right]$

for thin walled sections : $^*J \approx \frac{1}{3} \sum_{i=1}^{n} b_i t_i^3$

Back to stiffness derivations: k_{22}, k_{52}

$u_2 = \phi = 1.0 \qquad \phi = \frac{TL}{GJ} \qquad 1.0 = \frac{k_{22} L}{GJ} \quad \therefore \quad k_{22} = \frac{GJ}{L}$

from equilibrium $\sum M_x = 0 \qquad k_{22} + k_{52} = 0 \quad \therefore \quad k_{52} = -\frac{GJ}{L}$

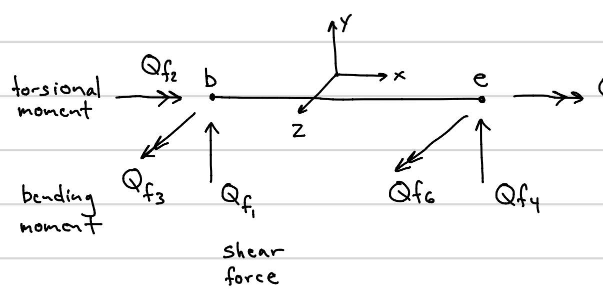
Similarly terms can be derived for $u_5 = 1.0$

$$[k]_{grid} = \frac{EI}{L^3} \begin{bmatrix} 12 & 0 & 6L & -12 & 0 & 6L \\ 0 & \frac{GJL^2}{EI} & 0 & 0 & -\frac{GJL^2}{EI} & 0 \\ 6L & 0 & 4L^2 & -6L & 0 & 2L^2 \\ -12 & 0 & -6L & 12 & 0 & -6L \\ 0 & -\frac{GJL^2}{EI} & 0 & 0 & \frac{GJL^2}{EI} & 0 \\ 6L & 0 & 2L^2 & -6L & 0 & 4L^2 \end{bmatrix}$$

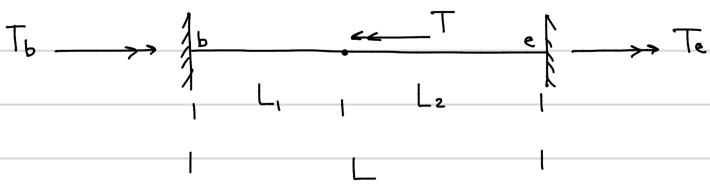
zeros in $[k]$ (i.e., sparsity) stems from bending/torsion being uncoupled

$$\{Q\} = \{Q_f\} + [k]\{u\}$$

$$\{Q_f\} = \begin{Bmatrix} FS_b \\ FT_b \\ FM_b \\ FS_e \\ FT_e \\ FM_e \end{Bmatrix} \begin{matrix} \text{shear force} \\ \text{torsional moment} \\ \text{bending moment} \\ \\ \\ \\ \end{matrix}$$



How to derive fixed-end forces for torsion?



Superposition

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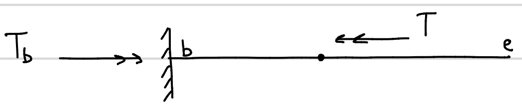
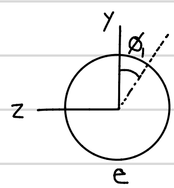


remove redundant, apply loading



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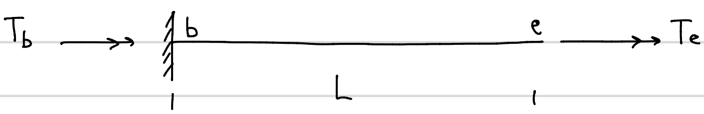
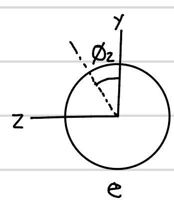


apply redundant reaction



Compatibility

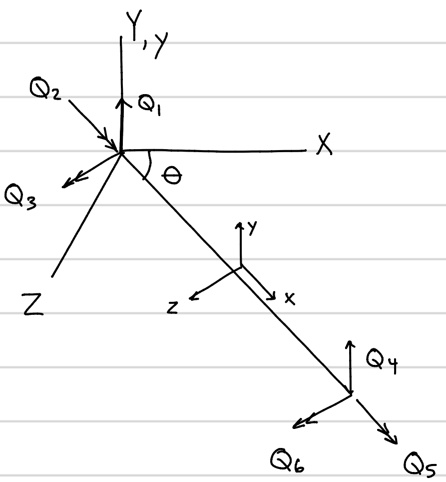
$$0 = \phi_1 + \phi_2 \qquad \phi_1 = -\frac{TL_1}{GJ} \qquad \phi_2 = \frac{T_e L}{GJ}$$

$$\therefore \quad T_e = T\frac{L_1}{L}$$

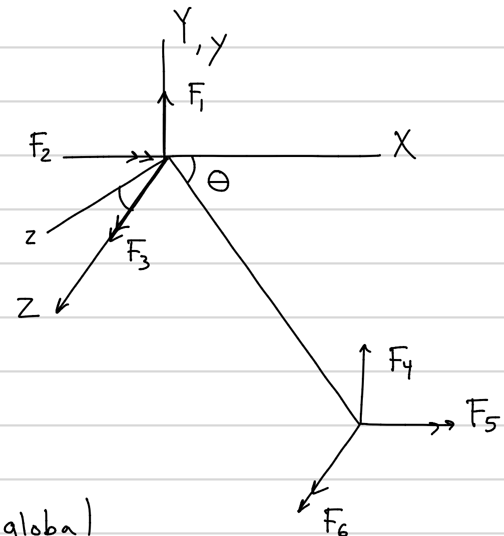
from equilibrium $\sum M_x = 0 \quad T_b - T + T_e = 0$

$$T_b = T - T\frac{L_1}{L} = \frac{T(L - L_1)}{L} \quad \therefore \quad T_b = T\frac{L_2}{L}$$

Coordinate Transformation



local



global

$$Q_1 = F_1 \qquad \text{(local } y \text{ and global } Y \text{ coincide)}$$

$$Q_2 = F_2 \cos\theta + F_3 \sin\theta$$

$$Q_3 = -F_2 \sin\theta + F_3 \cos\theta$$

similarly for $Q_4 - Q_6$ where $\{Q\} = [T]\{F\}$, $\{u\} = [T]\{v\}$

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C & S & 0 & 0 & 0 \\ 0 & -S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{bmatrix}$$

$$C = \cos\theta = \frac{L_x}{L}$$

$$^*\ S = \sin\theta = \frac{L_z}{L}\ ^* \qquad \underline{\underline{\text{NOT}}}\ \frac{L_y}{L}\ !$$

$$\{F\} = [T]^T\{Q\}, \quad \{v\} = [T]^T\{u\}$$

$$\{F\} = \underset{\displaystyle [T]^T\{Q_f\}}{\underset{\downarrow}{\{F_f\}}} + \underset{\displaystyle [T]^T[k][T]}{\underset{\downarrow}{[K]}}\{v\}$$