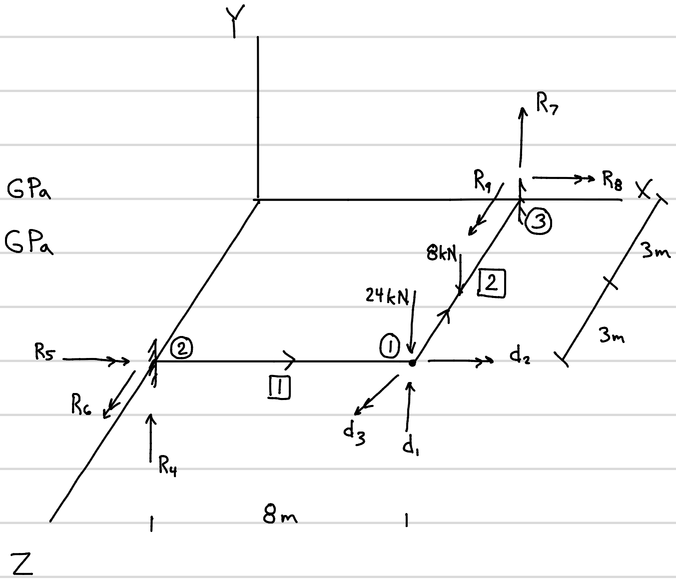
October 29, 2024
Matrix Displacement Method
Grids
Part II - Numerical Example

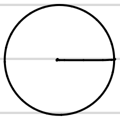
Grid Example

1. Label Structure



$E = 200$ GPa

$G = 76$ GPa



$r = 150$ mm

2. $$\{P\} = \begin{Bmatrix} -24 \\ 0 \\ 0 \end{Bmatrix} \text{ kN}$$

3. Member-level contributions

local

$$\{Qf\}' = \{0\}$$

$[k]$ $\frac{12EI}{L^3}$, $\frac{6EI}{L^2}$, $\frac{4EI}{L}$, $\frac{2EI}{L}$, $\frac{GJ}{L}$

$$I_{circle} = \frac{\pi r^4}{4} = 3.976e8 \text{ mm}^4$$

$$J_{circle} = \frac{\pi r^4}{2} = 7.952e8 \text{ mm}^4$$

$L^1 = 8000$ mm $\quad L^2 = 6000$ mm

$[k]^1$ $\quad \frac{12EI}{L^3} = 1.8638$ kN/mm $\qquad \frac{6EI}{L^2} = 7455.1$ kN

$\frac{4EI}{L} = 3.976e7$ kN·mm $\qquad \frac{2EI}{L} = 1.988e7$ kN·mm

$\frac{GJ}{L} = 7.555e6$ kN·mm

$$\{Qf\}^2 = \begin{Bmatrix} FS_b \\ FT_b \\ FM_b \\ FS_e \\ FT_e \\ FM_e \end{Bmatrix} \begin{matrix} \text{shear} \\ \text{torsion} \\ \text{bending} \\ \\ \\ \\ \end{matrix} = \begin{Bmatrix} 8/2 \\ 0 \\ \frac{8*6000}{8} \\ 8/2 \\ 0 \\ \frac{-8*6000}{8} \end{Bmatrix} = \begin{Bmatrix} 4 \\ 0 \\ 6000 \\ 4 \\ 0 \\ -6000 \end{Bmatrix} \begin{matrix} \text{kN} \\ \\ \text{kN·mm} \\ \\ \\ \\ \end{matrix}$$

$[k]^2$ $\quad \frac{12EI}{L^3} = 4.4179$ kN/mm $\qquad \frac{6EI}{L^2} = 13254$ kN

$\frac{4EI}{L} = 5.301e7$ kN·mm $\qquad \frac{2EI}{L} = 2.651e7$ kN·mm

$\frac{GJ}{L} = 1.007e7$ kN·mm

<u>global</u>

transformation

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & s & 0 & 0 & 0 \\ 0 & -s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & s \\ 0 & 0 & 0 & 0 & -s & c \end{bmatrix}$$

$c = \frac{L_x}{L} \qquad s = \frac{L_z}{L}$

$\boxed{1}$ $\quad c = 8/8 = 1 \qquad s = 0$

$\boxed{2}$ $\quad c = 0 \qquad s = -6/6 = -1$

$$\{F\} = \{F_f\} + [K]\{v\}$$

$$\{F_f\} = [T]^T\{Q_f\} \qquad \{F_f\}^1 = \{0\} \qquad \{F_f\}^2 = \begin{Bmatrix} 4 \\ 6000 \\ 0 \\ 4 \\ -6000 \\ 0 \end{Bmatrix}$$

$$[K] = [T]^T[k][T]$$

4. Assembly (code #)

	F_1, V_1	F_2, V_2	F_3, V_3	F_4, V_4	F_5, V_5	F_6, V_6
$\boxed{1}$	4	5	6	1	2	3
$\boxed{2}$	1	2	3	7	8	9

$$\underset{3\times1}{\{P_f\}} = \begin{Bmatrix} F_{f4}^1 + F_{f1}^2 \\ F_{f5}^1 + F_{f2}^2 \\ F_{f6}^1 + F_{f3}^2 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 6000 \\ 0 \end{Bmatrix} \begin{matrix} \text{kN} \\ \text{kN}\cdot\text{mm} \\ \end{matrix}$$

$$\underset{3\times3}{[S]} = \begin{bmatrix} K_{44}^1 + K_{11}^2 & K_{45}^1 + K_{12}^2 & K_{46}^1 + K_{13}^2 \\ K_{54}^1 + K_{21}^2 & K_{55}^1 + K_{22}^2 & K_{56}^1 + K_{23}^2 \\ K_{64}^1 + K_{31}^2 & K_{65}^1 + K_{32}^2 & K_{66}^1 + K_{33}^2 \end{bmatrix}$$

$$= \begin{bmatrix} 6.2817 & 13254 & -7455.1 \\ 13254 & 6.057e7 & 0 \\ -7455.1 & 0 & 4.9834e7 \end{bmatrix}$$

5. Solve

$$\{d\} = [S]^{-1}\{P - P_f\} = \begin{Bmatrix} -11.776 \\ 0.002478 \\ -0.001762 \end{Bmatrix} \begin{matrix} \text{mm} \\ \text{rad} \\ \text{rad} \end{matrix}$$

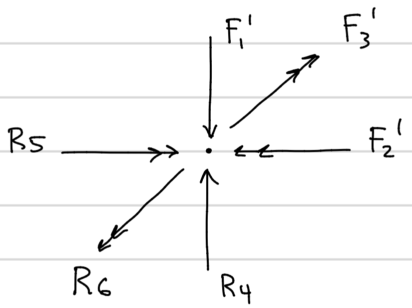
6. Post-process : compute member-end forces (global) compatibility d - v

$$\{F\} = \{F_f\} + [K]\{v\}$$

$$\{v\}^1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ d_1 \\ d_2 \\ d_3 \end{Bmatrix} \quad \{F\}^1 = \begin{Bmatrix} 8.81 \\ -18.72 \\ 52.77 \\ -8.81 \\ 18.72 \\ 17.75 \end{Bmatrix} \begin{matrix} \text{kN} \\ \text{kN}\cdot\text{m} \\ \text{kN}\cdot\text{m} \\ \text{kN} \\ \text{kN}\cdot\text{m} \\ \text{kN}\cdot\text{m} \end{matrix} \quad \{v\}^2 = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \{F\}^2 = \begin{Bmatrix} -15.19 \\ -18.72 \\ -17.75 \\ 23.19 \\ -96.40 \\ 17.75 \end{Bmatrix}$$

7. Support reactions

Joint 1



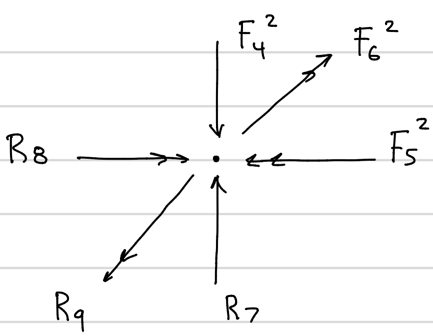
Equilibrium

$\sum F_y = 0 \quad R_4 = F_1^1 = 8.81$ kN

$\sum M_x = 0 \quad R_5 = F_2^1 = -18.72$ kN·m

$\sum M_z = 0 \quad R_6 = F_3^1 = 52.77$ kN·m

Joint 3



$\sum F_y = 0 \quad R_7 = F_4^2 = 23.19$ kN

$\sum M_x = 0 \quad R_8 = F_5^2 = -96.4$ kN·m

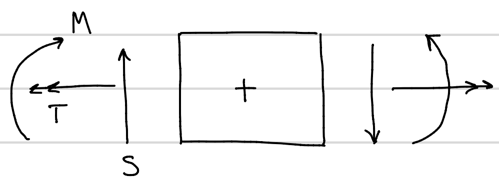
$\sum M_z = 0 \quad R_9 = F_6^2 = 17.75$ kN·m

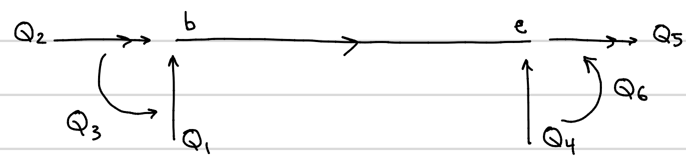
8. Draw torsion, shear, and bending moment diagrams

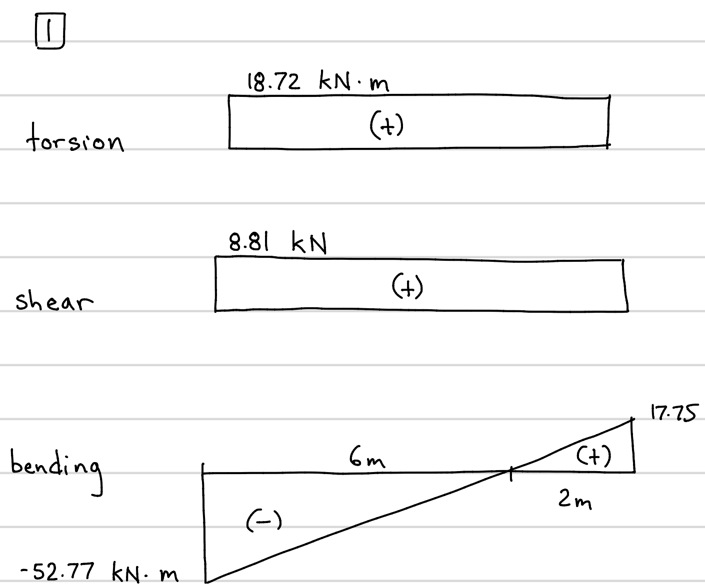
easier to work with $\{Q\} = [T]\{F\}$

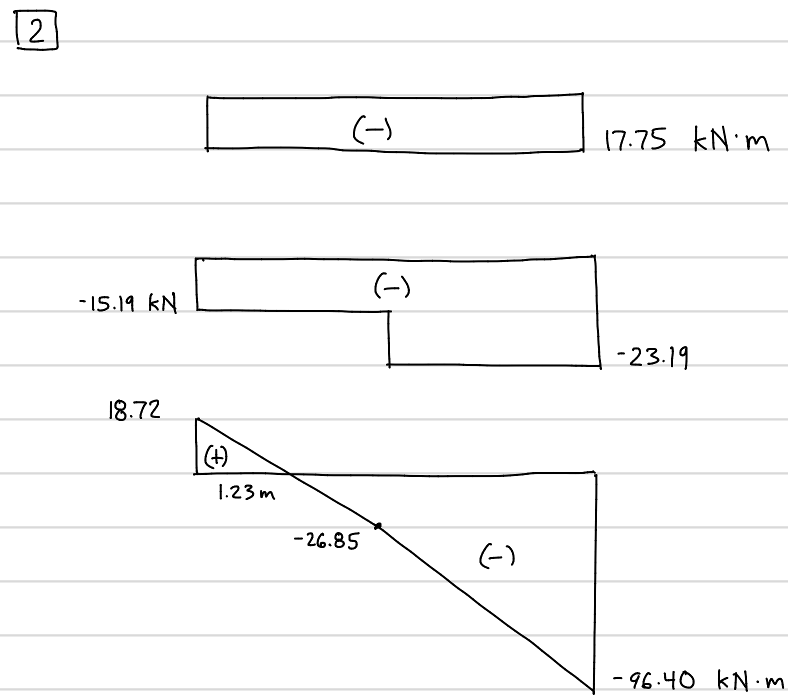
$$\{Q\}^1 = \left\{\begin{array}{c} 8.81 \\ -18.72 \\ 52.77 \\ -8.81 \\ 18.72 \\ 17.75 \end{array}\right\} \begin{array}{l} \text{shear} \\ \text{torsion} \\ \text{bending} \\ \text{kN} \\ \text{kN}\cdot\text{m} \\ \text{kN}\cdot\text{m} \end{array} \qquad \{Q\}^2 = \left\{\begin{array}{c} -15.19 \\ 17.75 \\ -18.72 \\ 23.19 \\ -17.75 \\ -96.4 \end{array}\right\} \begin{array}{l} \text{shear} \\ \text{torsion} \\ \text{bending} \\ \text{kN} \\ \text{kN}\cdot\text{m} \\ \text{kN}\cdot\text{m} \end{array}$$

sign convention









Calculate Stresses

Normal

$$\sigma_b = \frac{-My}{I} \quad \text{for circle } I = \frac{\pi r^4}{4} \quad \therefore \quad \sigma_b = \frac{-4M}{\pi r^3}$$

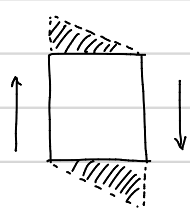
$$\boxed{1} \quad \sigma_b^1{}_{max} = \frac{-4(-52.77 * 1000)}{\pi(150^3)} = \underline{19.9 \text{ MPa}}$$

$$\boxed{2} \quad \sigma_b^2{}_{max} = \frac{-4(-96.40 * 1000)}{\pi(150^3)} = \underline{36.37 \text{ MPa}}$$

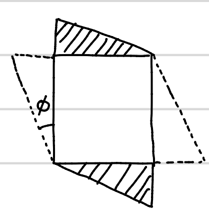
Shear

$$\tau_s = \frac{VQ}{Ib} \quad \text{for circle } \tau_s = \frac{4V}{3A} \quad \text{* valid at neutral axis *}$$

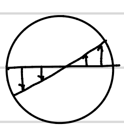
$$\tau_s^1 = \frac{4(8.81)}{3\pi(150^2)} = 0.167 \text{ MPa}$$



$$\tau_s^2 = \frac{4(23.19)}{3\pi(150^2)} = 0.437 \text{ MPa}$$



$$\tau_T = \frac{T\rho}{J} \quad \text{for circle}$$



$$\rho_{max} = r \qquad J = \frac{\pi r^4}{2}$$

$$\boxed{1} \quad \tau_T = \frac{(18.72 * 1000)(150)}{\frac{\pi(150^4)}{2}} = 3.53 \text{ MPa} \qquad \boxed{2} \quad \tau_T^2 = \frac{(17.75 * 1000)(150)}{\frac{\pi(150^4)}{2}} = 3.35 \text{ MPa}$$

$$\tau_{max}^1 = 0.167 + 3.53 = \underline{3.697 \text{ MPa}} \qquad \tau_{max}^2 = 0.437 + 3.35 = \underline{3.785 \text{ MPa}}$$