Linear Algebra Fundamentals

If I gave you the three (linear) equations:

$$3x_1 + 2x_2 + 1x_3 = 10$$
$$2x_1 + 4x_2 + 3x_3 = 19$$
$$1x_1 + 3x_2 + 5x_3 = 22$$

Could you solve these? How?

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 3 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 19 \\ 22 \end{Bmatrix} \qquad [A][x] = [B] \qquad [A]\{x\} = \{B\}$$

$$[A]_{mxn} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots \\ \dots & \dots & a_{ij} & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

m x n = rows by columns

In the preceding case:

[A] is a 3 x 3 "square matrix"
[x] is a 3 x 1 "column matrix/vector"
[B] is a 3 x 1 "column matrix/vector"

$$[A]_{2x3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad [B]_{3x2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

square matrix

$$\begin{bmatrix} a_{11} & \dots & \dots \\ \dots & a_{22} & \dots \\ \dots & \dots & a_{33} \end{bmatrix}$$

main **diagonal**

Transpose of a matrix: $\left[A_{ij}\right]^T = \left[A\right]_{ji}$

$$\left[A\right]^T = \begin{bmatrix} a_{11} & \underline{a_{21}} & \underline{a_{31}} \\ \underline{a_{12}} & a_{22} & \underline{a_{32}} \\ \underline{a_{13}} & \underline{a_{23}} & a_{33} \end{bmatrix}$$

Symmetric Matrix: $A = A^T$, i.e. $a_{ij} = a_{ji}$

$$\left[A\right] = \begin{bmatrix} 1 & 4 & -5 \\ 4 & 2 & 6 \\ -5 & 6 & 3 \end{bmatrix}$$

Diagonal Matrix: $a_{ij} = 0 \quad \text{for } i \neq j$

$$\left[A\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Identity Matrix: $a_{ij} = 1 \quad \text{for } i = j \ , \ a_{ij} = 0 \quad \text{for } i \neq j$

$$\left[I\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null Matrix: $a_{ij} = 0$

$$\left[O\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Equality $\left[A\right] = \left[B\right]$**:** If two matrices are of the same order (dimensions) and all of their elements are identical $a_{ij} = b_{ij}$

Matrix Operations

Addition and Subtraction: Only matrices of the same order (conformable) can be added/subtracted by operating on corresponding elements: $\left[A\right] + \left[B\right] = a_{ij} + b_{ij}$

$$\left[A\right] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \left[B\right] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \left[A+B\right] = \begin{bmatrix} 2 & 2 & 4 \\ 4 & 6 & 6 \end{bmatrix}$$

$$[A-B]=\begin{bmatrix} 0 & 2 & 2 \\ 4 & 4 & 6 \end{bmatrix} \qquad [B-A]=\begin{bmatrix} 0 & -2 & -2 \\ -4 & -4 & -6 \end{bmatrix}$$

Multiplication by a scalar: To multiply by a scalar, each element of the matrix must be multiplied by the scalar: $s[A]=s(a_{ij})$

$$[A]=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad s=2 \qquad s[A]=\begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

Multiplication of matrices: This can only be carried out if the number of columns of the first matrix [A] matches the number of rows in the second matrix [B], where the resulting matrix [C] has dimensions equal to the number of rows of the first matrix by the number of columns of the second matrix: $[A]_{mxn}[B]_{nxp}=[C]_{mxp}$

This can be carried out by algebraically summing each element of the i*th* row of [A] by the corresponding element of the j*th* row of [B]. $c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}$

$$[A]=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad [B]=\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \qquad [C]=[A][B]$$

$$[C]= \begin{matrix} (1\times1+2\times2+3\times3) & (1\times4+2\times5+3\times6) & (1\times7+2\times8+3\times9) \\ (4\times1+5\times2+6\times3) & (4\times4+5\times5+6\times6) & (4\times7+5\times8+6\times9) \end{matrix}$$

$$[C]=\begin{bmatrix} 14 & 32 & 50 \\ 32 & 77 & 122 \end{bmatrix}$$

**Note: $[A]$ pre-multiplies $[B]$; $[B][A]$ not possible!

Even when $[A][B]$ and $[B][A]$ is possible, generally: $[A][B] \neq [B][A]$

$$[A]=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, [B]=\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \qquad [A][B]=\begin{bmatrix} 11 & 5 \\ 25 & 11 \end{bmatrix} \qquad [B][A]=\begin{bmatrix} 6 & 10 \\ 10 & 16 \end{bmatrix}$$

It is therefore important to maintain the proper sequential order of matrices when computing matrix products.

Some common relations:

$$[A]([B]+[C]) = [A][B]+[A][C]$$

$$([A][B])^T = [B]^T[A]^T \qquad ([A][B][C])^T = [C]^T[B]^T[A]^T$$

Multiplication of any matrix $[A]$ by a conformable (same dimensions) ***null*** matrix $[O]$ yields a null matrix:

$$[A][O]=[O] \qquad [O][A]=[O]$$

Multiplication of any matrix $[A]$ by any conformable (same dimensions) ***identity*** matrix $[I]$ yields the original matrix:

$$[A][I]=[A] \qquad [I][A]=[A]$$

Inverse of a square matrix: The inverse is only defined for square matrices [A] as a matrix $[A]^{-1}$ where pre-multiplication of the original matrix by the inverse yields the identity matrix [I].

$$[A]^{-1}[A]=[I]$$

Thus for a system of linear equations as initially described $[A][x]=[B]$
The concept of an inverse is used to solve for the unknown variables:

$$[A]^{-1}[A][x]=[A]^{-1}[B] \qquad [I][x]=[A]^{-1}[B] \qquad [x]=[A]^{-1}[B]$$

In general, inverting a square matrix is computationally expensive and thus more economical solution techniques are employed for solving linear (matrix) systems of equations such as LU (lower-upper) factorization.

Orthogonality $\quad [Q]^{-1}=[Q]^T \qquad [Q]=\begin{bmatrix} 0.8 & 0.6 & 0 \\ -0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$