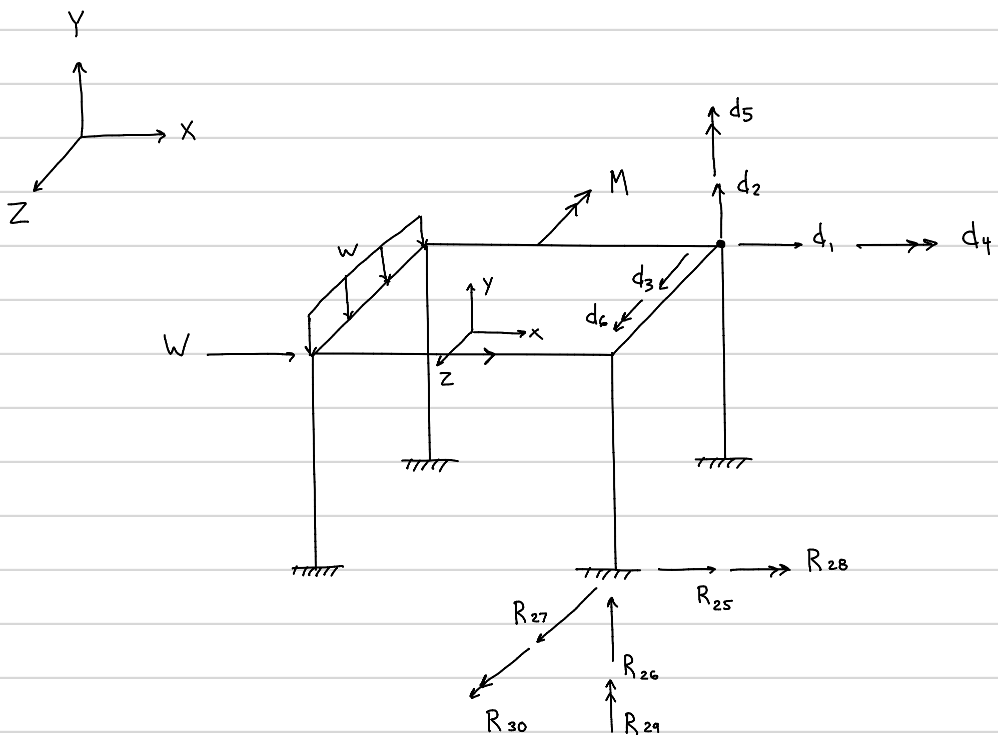
November 12, 2024
Matrix Displacement Method
Space Frames
Part I - Formulation

Space Frames

most general type of framed structure - under action of external loads, members are generally subjected to bending moments about principal axes, shears in both principal directions, torsional moments, and axial forces

Assumptions : member cross-sections are

1. symmetric about at least two mutually perpendicular axes
2. free to warp out of plane under torsion (i.e. uncoupled bending-torsional stiffness)



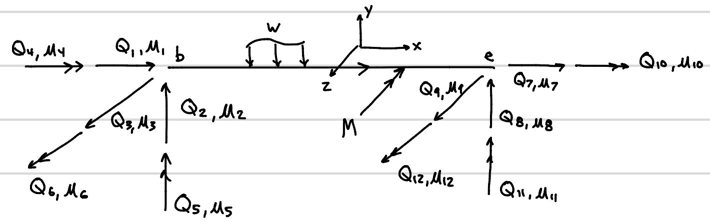
6 DOFs per joint
- 3 translations XYZ
- 3 rotations XYZ

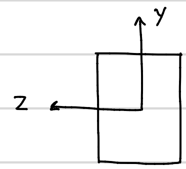
6 reactions
- 3 forces XYZ
- 3 moments XYZ

Total Structure : 24 DOFs, 24 Reactions

$$\{P\} = \{P_f\} + [S]\{d\}$$

member-level (local)





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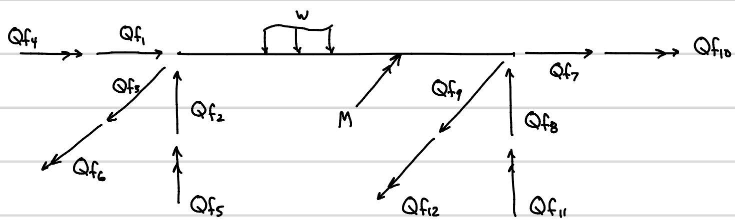
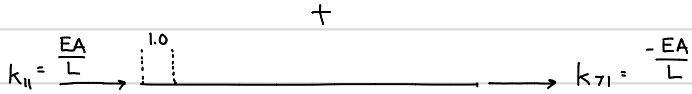


Apply loading

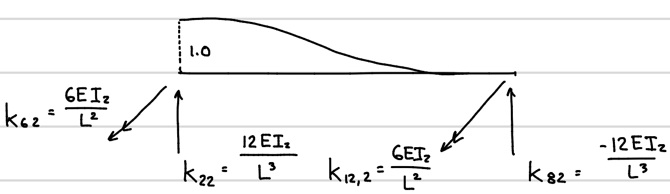
$u_1 - u_{12} = 0$

+



$u_1 = 1.0$, rest $= 0$

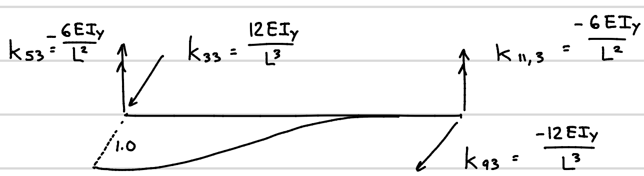
+



$u_2 = 1.0$, rest $= 0$

+

$u_3 = 1.0$, rest $= 0$



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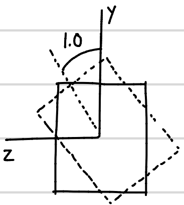
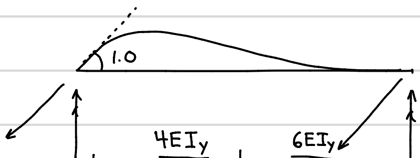


$k_{44} = \frac{GJ}{L}$ $\qquad$ $k_{10,4} = \frac{-GJ}{L}$ $\qquad$ $u_4 = 1.0$, rest $= 0$

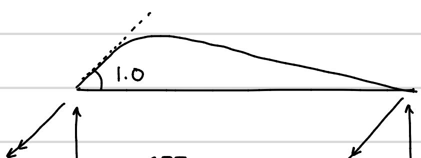
$+$



$u_5 = 1.0$, rest $= 0$

$k_{35} = \frac{-6EI_y}{L^2}$ $\qquad$ $k_{55} = \frac{4EI_y}{L}$ $\qquad$ $k_{95} = \frac{6EI_y}{L^2}$ $\qquad$ $k_{11,5} = \frac{2EI_y}{L}$

$+$



$u_6 = 1.0$, rest $= 0$

$k_{66} = \frac{4EI_z}{L}$ $\qquad$ $k_{26} = \frac{6EI_z}{L^2}$ $\qquad$ $k_{12,6} = \frac{2EI_z}{L}$ $\qquad$ $k_{86} = \frac{-6EI_z}{L^2}$

$+$

Similarly stiffness terms for $u_7 - u_{12}$ can be determined to complete 12×12 local stiffness matrix $[k]$

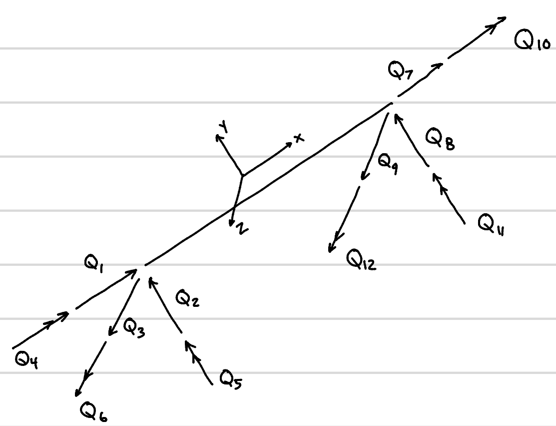
$$[k] = \frac{E}{L^3}\begin{bmatrix}
AL^2 & 0 & 0 & 0 & 0 & 0 & -AL^2 & 0 & 0 & 0 & 0 & 0 \\
 & 12I_z & 0 & 0 & 0 & 6LI_z & 0 & -12I_z & 0 & 0 & 0 & 6LI_z \\
 & & 12I_y & 0 & -6LI_y & 0 & 0 & 0 & -12I_y & 0 & -6LI_y & 0 \\
 & & & \frac{GJL^2}{E} & 0 & 0 & 0 & 0 & 0 & \frac{-GJL^2}{E} & 0 & 0 \\
 & & & & 4L^2I_y & 0 & 0 & 0 & 6LI_y & 0 & 2L^2I_y & 0 \\
 & & & & & 4L^2I_z & 0 & -6LI_z & 0 & 0 & 0 & 2L^2I_z \\
 & \text{Symmetric} & & & & & AL^2 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & 12I_z & 0 & 0 & 0 & -6LI_z \\
 & & & & & & & & 12I_y & 0 & 6LI_y & 0 \\
 & & & & & & & & & \frac{GJL^2}{E} & 0 & 0 \\
 & & & & & & & & & & 4L^2I_y & 0 \\
 & & & & & & & & & & & 4L^2I_z
\end{bmatrix}$$

$$\{Q\} = \{Qf\} + [k]\{u\}$$

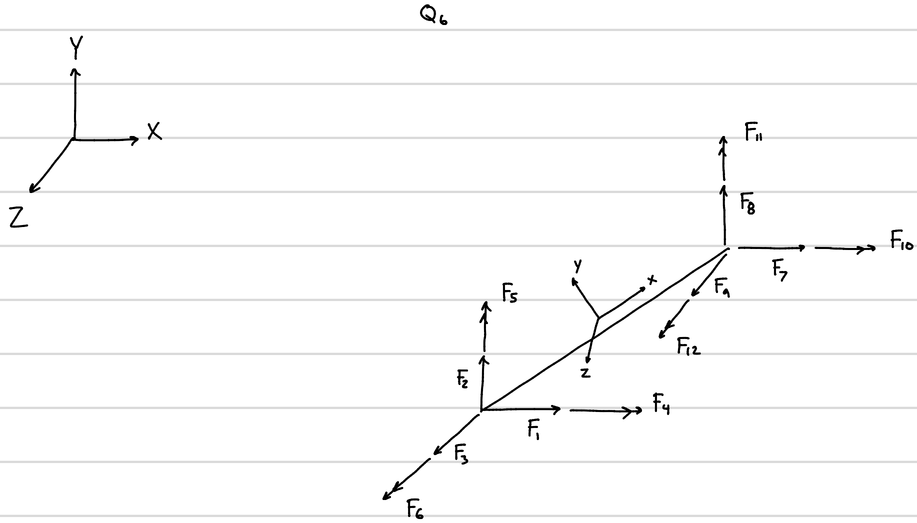
$$\underset{12\times1}{\{Qf\}} = \left\{ \begin{array}{c} \begin{bmatrix} FA_b \\ FS_{by} \\ FS_{bz} \\ FT_b \\ FM_{by} \\ FM_{bz} \end{bmatrix} \\ \hline 6\times1 \text{ end} \end{array} \right\} \quad \begin{array}{l} \text{axial force in local x} \\ \text{shear force in local y} \\ \text{shear force in local z} \\ \text{torsional moment about local x} \\ \text{bending moment about local y} \\ \text{bending moment about local z} \end{array}$$

Can be computed from previously derived and/or tabulated values

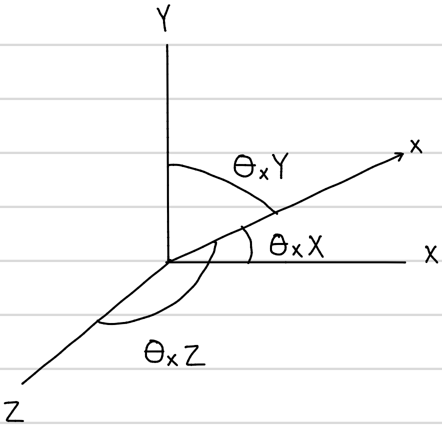
Coordinate Transformations



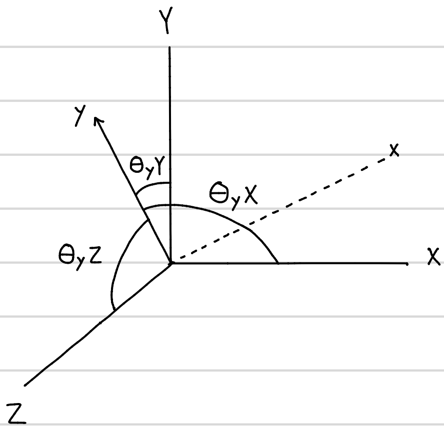
local



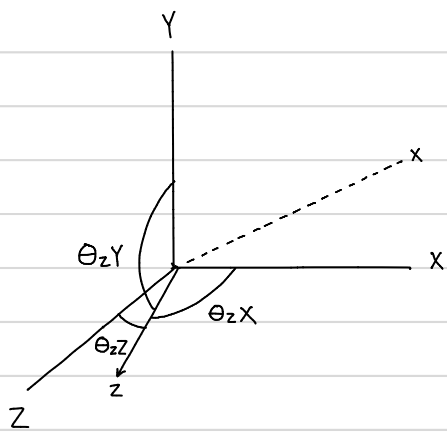
global



local x-axis orientation



local y-axis orientation



local z-axis orientation

$$Q_1 = F_1 \cos\theta_x X + F_2 \cos\theta_x Y + F_3 \cos\theta_x Z$$
$$Q_2 = F_1 \cos\theta_y X + F_2 \cos\theta_y Y + F_3 \cos\theta_y Z$$
$$Q_3 = F_1 \cos\theta_z X + F_2 \cos\theta_z Y + F_3 \cos\theta_z Z$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{bmatrix} r_x X & r_x Y & r_x Z \\ r_y X & r_y Y & r_y Z \\ r_z X & r_z Y & r_z Z \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$r_{iJ} = \cos\theta_{iJ}$, $i = x, y, z$ $J = X, Y, Z$

$r \equiv$ member rotation matrix

similarly

$$\begin{Bmatrix} Q_{4,7,10} \\ Q_{5,8,11} \\ Q_{6,9,12} \end{Bmatrix} = \begin{bmatrix} r_xX & r_xY & r_xZ \\ r_yX & r_yY & r_yZ \\ r_zX & r_zY & r_zZ \end{bmatrix} \begin{Bmatrix} F_{4,7,10} \\ F_{5,8,11} \\ F_{6,9,12} \end{Bmatrix}$$

where $\underset{12\times1}{\{Q\}} = \underset{12\times12}{[T]}\underset{12\times1}{\{F\}} \quad \left(\{u\} = [T]\{v\} \right)$

$$[T] = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \end{bmatrix}$$

r = 3x3 member rotation matrix

0 = 3x3 null matrix

also $\{F\} = [T]^T\{Q\}$ $\quad \{F_f\} = [T]^T\{Q_f\}$

$\{v\} = [T]^T\{u\}$ $\quad [K] = [T]^T[k][T]$

$$\{F\} = \{F_f\} + [K]\{v\}$$

* Member rotation matrix r consists of 9 elements

First row is easily computed with coordinates of member endpoints

$$r_xX = \cos\theta_xX = \frac{X_e - X_b}{L}$$

$$r_xY = \cos\theta_xY = \frac{Y_e - Y_b}{L}$$

$$r_xZ = \cos\theta_xZ = \frac{Z_e - Z_b}{L}$$

$$L = \sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2 + (Z_e - Z_b)^2}$$