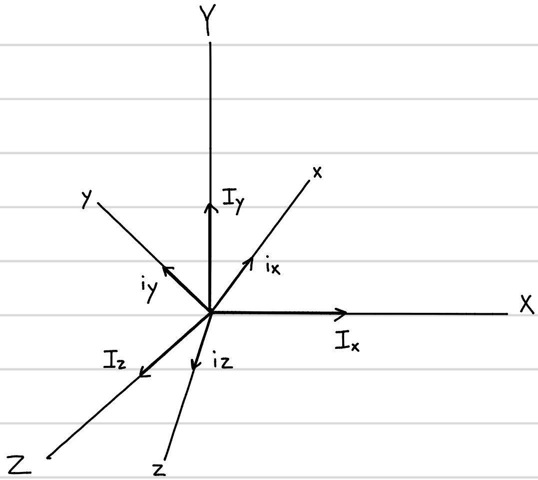
November 14, 2024
Matrix Displacement Method
Space Frames
Part II - Transformation



i_x, i_y, i_z unit vectors in directions of local axes

I_x, I_y, I_z unit vectors in directions of global axes

$$\begin{Bmatrix} i_x \\ i_y \\ i_z \end{Bmatrix} = \underbrace{\begin{bmatrix} r_xX & r_xY & r_xZ \\ r_yX & r_yY & r_yZ \\ r_zX & r_zY & r_zZ \end{bmatrix}}_{[r]} \begin{Bmatrix} I_x \\ I_y \\ I_z \end{Bmatrix}$$

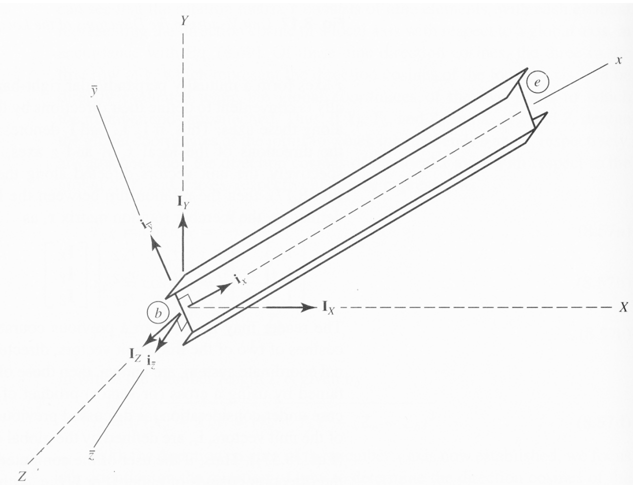
If we know direction cosines of 2/3 unit vectors
the third can be determined by using cross (vector) product

i_x known from coordinates : $r_xX = \frac{X_e - X_b}{L}$, $r_xY = \frac{Y_e - Y_b}{L}$, $r_xZ = \frac{Z_e - Z_b}{L}$ $\quad L = \sqrt{(X_e - X_b)^2 + (Y_e - Y_b)^2 + (Z_e - Z_b)^2}$

calculation of i_y and i_z is rather tedious and thus another approach
is typically pursued based on roll angle (ψ)

Derivation of [r] is a two-step process :

1st step : member's x-axis is oriented in desired direction (x), where y and z axes are oriented
such that xy plane is vertical and z-axis lies in horizontal plane



(a) Member Orientation with $x\bar{y}$ Plane Vertical

Fig. 8.18

$\bar{z} = i_x \times I_y$

aside: $\quad a \times b = \det \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \det \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \det \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \det \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$

$$= (a_2 b_3 - a_3 b_2)\, i - (a_1 b_3 - a_3 b_1)\, j + (a_1 b_2 - a_2 b_1)\, k$$

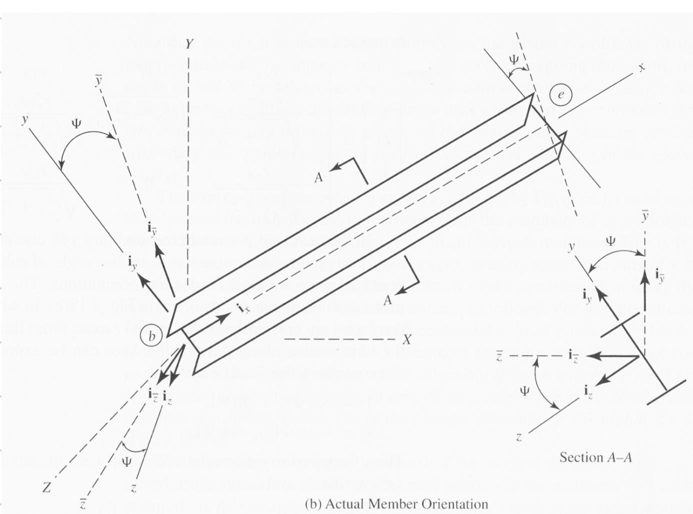
$$\bar{Z} = i_x \times I_y = \det \begin{vmatrix} I_x & I_y & I_z \\ r_x X & r_x Y & r_x Z \\ 0 & 1 & 0 \end{vmatrix} = -r_x Z\, I_x + r_x X\, I_z$$

$$i_{\bar{z}} = \frac{\bar{Z}}{|\bar{Z}|} = \frac{-r_x Z\, I_x + r_x X\, I_z}{\sqrt{r_x Z^2 + r_x X^2}}$$

$$i_{\bar{y}} = i_{\bar{z}} \times i_x = \frac{1}{\sqrt{r_x X^2 + r_x Z^2}} \left(-r_x X\, r_x Y\, I_x + I_y - r_x Y\, r_x Z\, I_z \right)$$

$$* \quad \begin{Bmatrix} i_x \\ i_{\bar{y}} \\ i_{\bar{z}} \end{Bmatrix} = \begin{vmatrix} r_x X & r_x Y & r_x Z \\ -r_x X\, r_x Y / |\bar{Z}| & |\bar{Z}| & -r_x Y\, r_x Z / |\bar{Z}| \\ -r_x Z / |\bar{Z}| & 0 & r_x X / |\bar{Z}| \end{vmatrix} \begin{Bmatrix} I_x \\ I_y \\ I_z \end{Bmatrix}$$

2nd Step: rotate auxiliary $x \bar{y} \bar{z}$ coordinate system about its local x-axis, counterclockwise by angle of roll ψ, until the member's principal axes are in desired orientations xyz



(b) Actual Member Orientation

Fig. 8.18 *(continued)*

$$i_y = \cos\psi\, i_{\bar{y}} + \sin\psi\, i_{\bar{z}}$$
$$i_z = -\sin\psi\, i_{\bar{y}} + \cos\psi\, i_{\bar{z}}$$

$$\begin{Bmatrix} i_x \\ i_y \\ i_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \begin{Bmatrix} i_x \\ i_{\bar{y}} \\ i_{\bar{z}} \end{Bmatrix}$$

substitute resulting equations * from step # 1

$$\begin{Bmatrix} i_x \\ i_y \\ i_z \end{Bmatrix} = \begin{vmatrix} r_xX & r_xY & r_xZ \\ (-r_xX\,r_xY\cos\psi - r_xZ\sin\psi)/|\bar{Z}| & |\bar{Z}|\cos\psi & (-r_xY\,r_xZ\cos\psi + r_xX\sin\psi)/|\bar{Z}| \\ (r_xX\,r_xY\sin\psi - r_xZ\cos\psi)/|\bar{Z}| & -|\bar{Z}|\sin\psi & (r_xY\,r_xZ\sin\psi + r_xX\cos\psi)/|\bar{Z}| \end{vmatrix} \begin{Bmatrix} I_x \\ I_y \\ I_z \end{Bmatrix}$$

$$\begin{Bmatrix} i_x \\ i_y \\ i_z \end{Bmatrix} = \begin{bmatrix} & & \\ & r & \\ & & \end{bmatrix} \begin{Bmatrix} I_x \\ I_y \\ I_z \end{Bmatrix} \qquad |\bar{Z}| = \sqrt{r_xX^2 + r_xZ^2}$$

rotation matrix depends only on global coordinates of member endpoints and angle of roll $\psi \equiv$ angle measured clockwise positive when looking in negative x-dir through which local xyz coordinate system must be rotated about x-axis so that xy plane becomes vertical with local y-axis in the positive direction of global Y-axis.

Above can be used to determine $[T]$ for any member of a space frame EXCEPT vertical members, r_xX and $r_xZ = 0$ causing some elements of r to be undefined

Alternate definition for vertical member $\psi \equiv$ angle measured clockwise when looking in negative x-dir through which xyz coordinate system must be rotated about x-axis so that local z axis becomes parallel to and points in positive direction of global Z axis

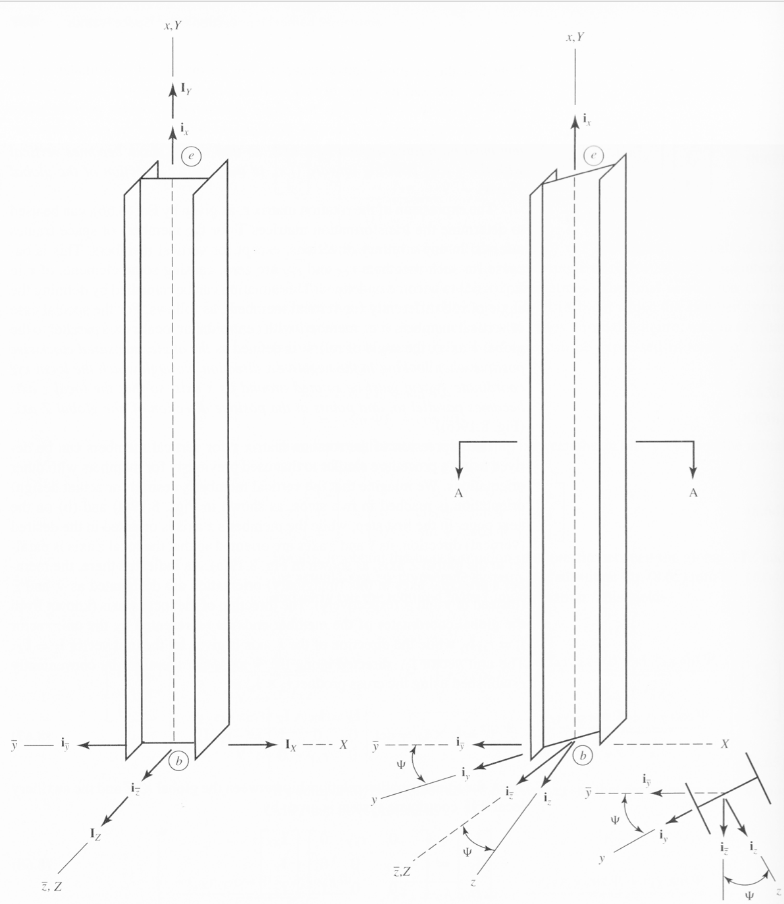
Step 1 : $\quad i_x = r_xY\, I_y \quad (r_xX \text{ and } r_xZ = 0)$

$$\bar{z} = i_{\bar{z}} = I_z$$

$$i_{\bar{y}} = i_{\bar{z}} \times i_x = \det \begin{vmatrix} I_x & I_y & I_z \\ 0 & 0 & 1 \\ 0 & r_xY & 0 \end{vmatrix} = -r_xY\, I_x$$

$$\begin{Bmatrix} i_x \\ i_{\bar{y}} \\ i_{\bar{z}} \end{Bmatrix} = \begin{bmatrix} 0 & r_x Y & 0 \\ -r_x Y & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} I_x \\ I_y \\ I_z \end{Bmatrix} \quad **$$

step 2 : rotate auxiliary $x\bar{y}\bar{z}$ coordinate system about its x-axis counterclockwise by angle of roll ψ until the member's principal axes are in desired orientations



(a) Orientation of Vertical Member with $\bar{z}$ Axis Parallel to Global Z Axis

(b) Actual Orientation of Vertical Member

Fig. 8.19

$$\begin{Bmatrix} i_x \\ i_y \\ i_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \begin{Bmatrix} i_x \\ i_{\bar{y}} \\ i_{\bar{z}} \end{Bmatrix}$$

substitute resulting equations ** from step # 1

$$\begin{Bmatrix} i_x \\ i_y \\ i_z \end{Bmatrix} = \begin{bmatrix} 0 & r_x Y & 0 \\ -r_x Y\cos\psi & 0 & \sin\psi \\ r_x Y\sin\psi & 0 & \cos\psi \end{bmatrix} \begin{Bmatrix} I_x \\ I_y \\ I_z \end{Bmatrix}$$

$\downarrow$ $\underset{3\times 3}{[r]}$ (for vertical members)

$$\underset{12\times 12}{[T]} = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \end{bmatrix}$$