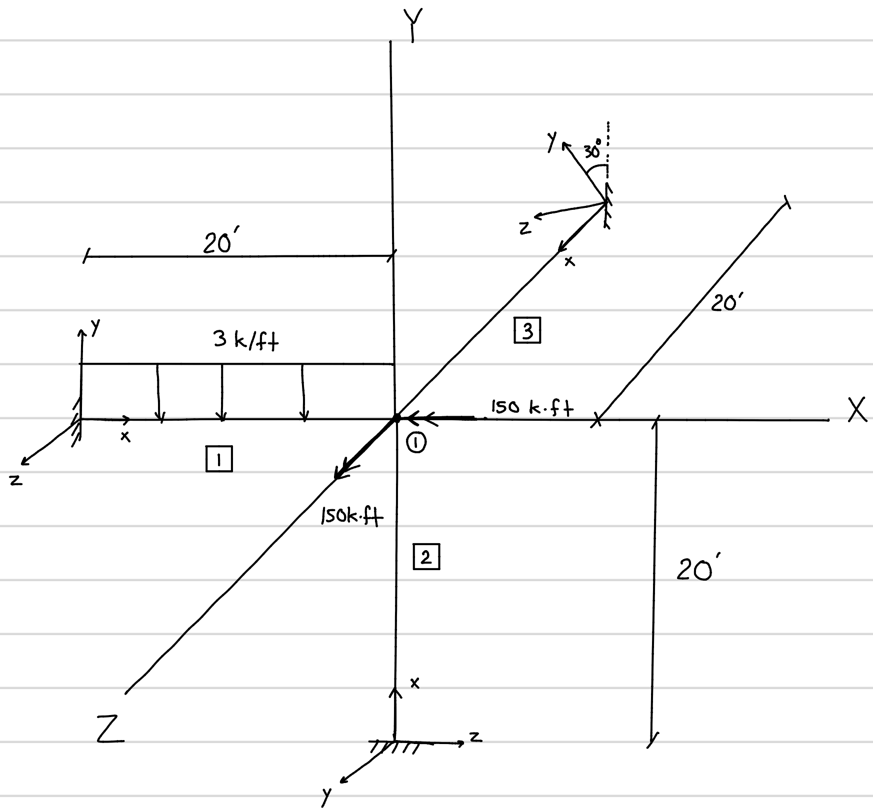
November 19, 2024
Matrix Displacement Method
Space Frames
Part III - Numerical Example

Space Frame Example

1. Label structure



$E = 29{,}000$ ksi

$G = 11{,}500$ ksi

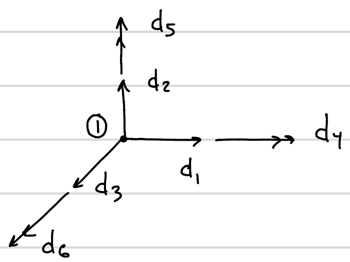
$A = 32.9$ in^2

$I_z = 716$ in^4

$I_y = 236$ in^4

$J = 15.1$ in^4

2. create $\{P\}$



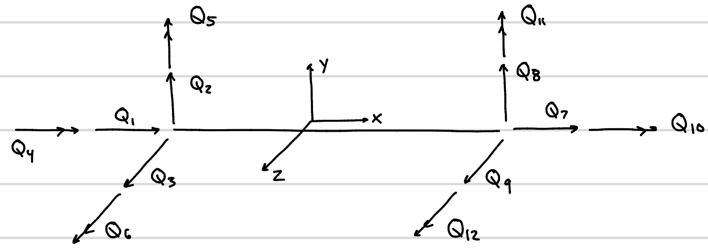
$\{P\}_{6\times1}$

$P_4 = -150 * 12 = -1800$ k·in

$P_6 = 150 * 12 = 1800$ k·in

3. member-level contributions

(local)



$\{Q_f\}^{2,3} = \{0\}$ $\quad$ $\{Q_f\}^{1}_{12\times1}$ $\quad$ uniform load : $\quad$ shear forces − Q_{f2}, Q_{f8} $\quad$ $\frac{wL}{2} = 30$k $\quad$ Q_{f2} (+) $\quad$ Q_{f8} (+)

bending moments − Q_{f6}, Q_{f12} $\quad$ $\frac{wL^2}{12} = 1200$ k·in $\quad$ Q_{f6} (+) $\quad$ Q_{f12} (−)

$[k]_{12\times12}$ calculated from material / geometric properties : EA/L, GJ/L, $12EI/L^3$, $6EI/L^2$, $4EI/L$, $2EI/L$ $\underline{I}$: I_y, I_z

(global) $\{F_f\} = [T]^T \{Q_f\}$ $[K] = [T]^T [k] [T]$

$$[T]_{12\times12} = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \end{bmatrix}$$ where $[r]_{3\times3}$ member rotation matrix

$\boxed{1}$ horizontal member w/ local axes oriented along global axes

$\therefore$ $[r]^1 = [I]$, $[T]^1 = [I]$

$\boxed{2}$ vertical member

$$[r] = \begin{bmatrix} 0 & r_xY & 0 \\ -r_xY\cos\psi & 0 & \sin\psi \\ r_xY\sin\psi & 0 & \cos\psi \end{bmatrix}$$

$$r_xX = \frac{X_e - X_b}{L} \equiv \frac{L_x}{L}$$
$$r_xY = \frac{Y_e - Y_b}{L} \equiv \frac{L_y}{L}$$
$$r_xZ = \frac{Z_e - Z_b}{L} \equiv \frac{L_z}{L}$$

$$L = \sqrt{L_x^2 + L_y^2 + L_z^2}$$

$X_e - X_b = 0 - 0$, $Y_e - Y_b = 0 - (-20)$, $Z_e - Z_b = 0 - 0$ $L = 20$ ft.   * watch units in $[k]$

$\downarrow$

$r_xY = \frac{20}{20} = 1$

$\psi \equiv$ roll angle : looking in negative x-direction, angle measured clockwise rotating about local x-axis such that local z-axis points/aligns with global z-axis

$\psi = 90°$

$\cos 90 = 0$

$\sin 90 = 1$

$$[r]^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$\boxed{3}$ arbitrarily oriented member

$$[r] = \begin{vmatrix} r_xX & r_xY & r_xZ \\ (-r_xX\,r_xY\cos\psi - r_xZ\sin\psi)/|\bar{Z}| & |\bar{Z}|\cos\psi & (-r_xY\,r_xZ\cos\psi + r_xX\sin\psi)/|\bar{Z}| \\ (r_xX\,r_xY\sin\psi - r_xZ\cos\psi)/|\bar{Z}| & -|\bar{Z}|\sin\psi & (r_xY\,r_xZ\sin\psi + r_xX\cos\psi)/|\bar{Z}| \end{vmatrix}$$

$$|\bar{Z}| = \sqrt{r_xZ^2 + r_xX^2}$$

$X_e - X_b = 0 - 0$, $Y_e - Y_b = 0 - 0$, $Z_e - Z_b = 0 - (-20)$ $L = 20$ ft.

$r_xX = 0$ $r_xY = 0$ $r_xZ = \frac{20}{20} = 1$ $\therefore$ $|\bar{Z}| = 1$

$\psi \equiv$ roll angle looking in negative x-direction, angle measured clockwise rotating about local x-axis such that local xy plane becomes vertical with local y-axis points/aligns with global Y-axis

$\psi = 30^\circ$

$\cos 30 = \sqrt{3}/2$

$\sin 30 = 1/2$

$$[r]^3 = \begin{bmatrix} 0 & 0 & 1 \\ -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \end{bmatrix}$$

Compute global member contributions $\{F_f\} = [T]^T\{Q_f\}$ $[K] = [T]^T[k][T]$

4. Assembly via Code # $\{F_f\} \rightarrow \{P_f\}$ $[K] \rightarrow [S]$

$\boxed{1}$	7	8	9	10	11	12	1	2	3	4	5	6
$\boxed{2}$	13	14	15	16	17	18	1	2	3	4	5	6
$\boxed{3}$	19	20	21	22	23	24	1	2	3	4	5	6

$$\{P_f\} = \begin{Bmatrix} 0 \\ 30 \\ 0 \\ 0 \\ 0 \\ -1200 \end{Bmatrix} \begin{matrix} \\ k \\ \\ \\ \\ k\cdot in \end{matrix} \qquad [S] = \begin{bmatrix} 3990.3 & -5.23 & 0 & -627.87 & -1075.4 & 712.92 \\ & 4008.4 & 0 & 1800.4 & 627.87 & -2162.9 \\ & & 3999.4 & -2162.9 & 712.92 & 0 \\ & \text{Symmetric} & & 634857 & 100459 & 0 \\ & & & & 286857 & 0 \\ & & & & & 460857 \end{bmatrix}$$

5. Solve $\{P - P_f\} = [S]\{d\}$

$$\{d\} = \begin{Bmatrix} -1.3522 \\ -2.7965 \\ -1.812 \\ -3.0021 \\ 1.0569 \\ 6.4986 \end{Bmatrix} \times 10^{-3} \begin{matrix} \text{in.} \\ \text{in.} \\ \text{in.} \\ \text{rad.} \\ \text{rad.} \\ \text{rad.} \end{matrix}$$

$\boxed{\text{Post-process}}$

6. Member end forces

Option 1 : $\{F\} = \{F_f\} + [K]\{v\}$
(global)

$\{v\}$ → compatibility w/ $\{d\}$

Option 2 : $\{Q\} = \{Q_f\} + [k]\{u\}$
(local)

$\{u\}$ → $[T]\{v\}$

$\{v\}$ → compatibility w/ $\{d\}$

7. Compute support reactions (global) - joint equilibrium w/ $\{F\}$

For option 1 : $\{F\} \rightarrow \{R\}$ direct ; option 2 : $\{F\} = [T]^T\{Q\} \longrightarrow \{R\}$

8. Draw axial force, shear force (x 2), torsional moment, and bending moment (x 2) diagrams * $\{Q\}$ easiest

9. Calculate normal stresses (axial force + bending moment) $\sigma_a = \frac{N}{A}$ $\sigma_b = -\frac{My}{I}$

$\sigma_{max} = \sigma_a + \sigma_b$

10. Calculate shear stresses (shear force + torsional moment) $\tau_s = \frac{VQ}{Ib}$ $\tau_T = \frac{Tp}{J}$

$\tau_{max} = \tau_s + \tau_T$