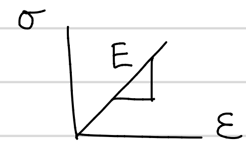
November 21, 2024
Matrix Displacement Method
Nonlinear Analysis
Part I - Intro, solution strategy

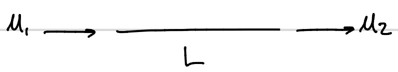
Nonlinear Structural Analysis

Up to this point, focused on linear analysis, based on two fundamental assumptions:

1. material linearity, i.e. linear elastic constitutive relation $\sigma = E\varepsilon$

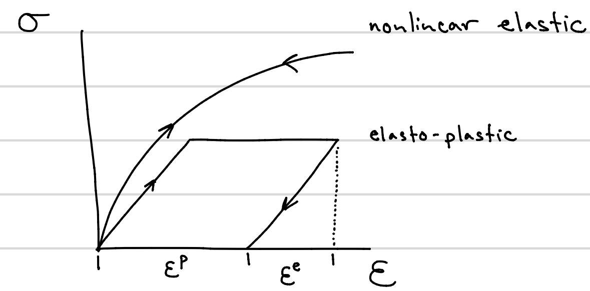


2. geometric linearity, i.e. small enough deformations such that member strains can be expressed as linear functions of joint displacements $\left(\varepsilon_{bar} = \frac{u_2 - u_1}{L}\right)$ and thus the equilibrium equations can be written about undeformed geometry

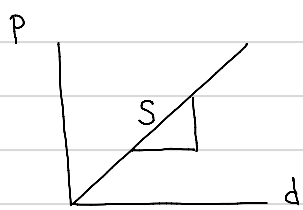


Nonlinear analysis needed to accurately predict:

(material) plasticity, nonlinear elastic response, damage
(geometric) large (finite) deformations, instability



single DOF system (scalar function)



$P = Sd$

$P = f(d)$

linear

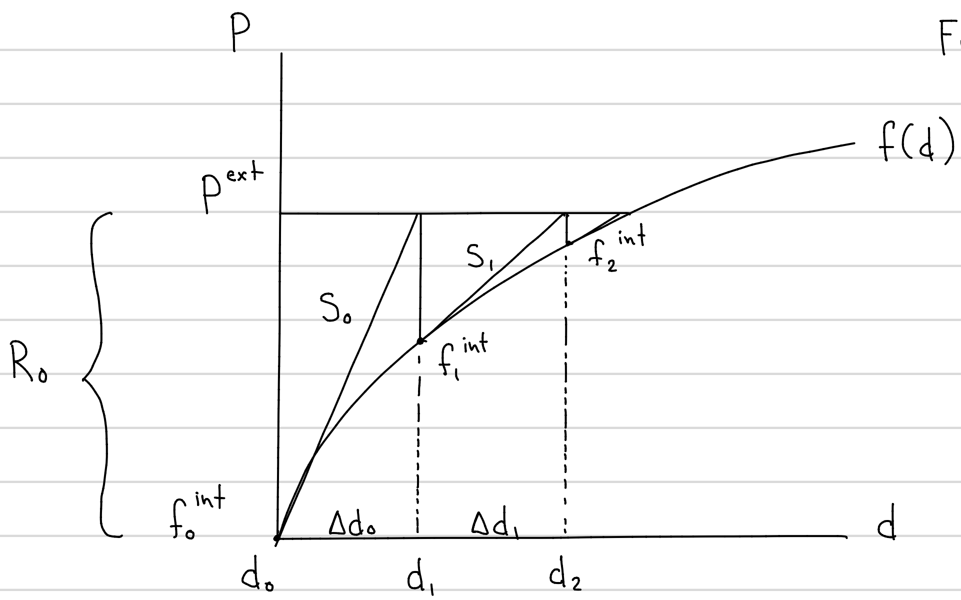
1. d-given, can find $P = Sd$
2. P-given, can find $d = \frac{1}{S} P$

nonlinear

1. d-given, can find $P = f(d)$
2. P-given, can NOT find d, $f^{-1}(d)$ not known

1. displacement - controlled
2. force-controlled (typical of real-world problems)

Solving single and multi DOF (SDOF/MDOF) nonlinear equations requires an iterative procedure : Newton Raphson Method (based on a first-order Taylor series expansion, linearized equations)



Find d for given P

$$P = f(d)$$

$$P^{ext} = f^{int}$$

$P^{ext} - f^{int} = 0$ equilibrium

$\neq 0$ unbalanced

≈ 0 converged

iteration i : $d_{i+1} = d_i + \Delta d_i$

$$\Delta d_i = \frac{P^{ext} - f^{int}(d_i)}{S_{tan}(d_i)}$$

** issues arise when

$S_{tan} = 0$ (SDOF)

$[S]_{tan}$ = singular (MDOF)

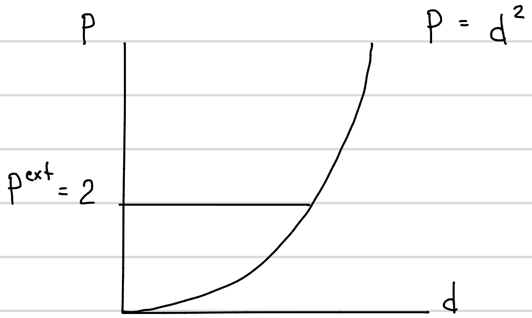
$$S_{tan} = \frac{\partial f(d_i)}{\partial d_i}$$

Convergence criteria based on unbalanced force, residual : $R \equiv |P^{ext} - f^{int}|$

continue iterations until $R_i < \varepsilon R_0$, ε user defined tolerance, e.g. 10^{-6}

Iterative procedure can be performed in multiple load steps (n) to generate "entire curve"

Example:



$\varepsilon = 0.001$

$$P^{ext} = 2$$
$$f^{int} = d^2$$
$$S_t = \frac{df^{int}}{dd} = 2d$$

$(f^{-1} = \sqrt{P})$

$$\Delta d = \frac{P^{ext} - f^{int}}{S_t} \qquad d_{i+1} = d_i + \Delta d_i$$

1st iteration: $d_0 = 0$ $\quad S_t = 2(0) = 0$ $\quad \Delta d = \dfrac{2-(0)^2}{0} = \infty$

* use caution w/ nonlinear analysis, certain "functions" require specialized solution strategies/solvers: e.g. arc-length method

remedy here: change initial guess, i.e. $d_0 = 1.0$

$S_t = 2(1.0) = 2.0$ $\quad \Delta d_0 = \dfrac{2-(1.0)^2}{2.0} = 0.5$ $\quad d_1 = 1.0 + 0.5 = 1.5$

Check residual: $R = |P^{ext} - f^{int}| = |2-(1.5)^2| = 0.25$ $\quad$ * Convergence Criteria $R_i < \varepsilon R_0$ *

$\longrightarrow$ <u>initialize</u> $R_0 = |P^{ext} - f^{int}(d_0)| = |2-(1.0)^2| = 1.0$ $\quad 0.25 > 0.001(1.0)$

2nd iteration: $d_1 = 1.5$ $\quad S_t = 2(1.5) = 3.0$ $\quad \Delta d = \dfrac{2-(1.5)^2}{3.0} = -1/12$

$d_2 = 1.5 - 1/12 = 1.41\bar{6}$ $\quad R_2 = |2-(1.41\bar{6})^2| = 0.00694 > 0.001\ (1.0)$

3rd iteration: $d_2 = 1.41\bar{6}$ $\quad S_t = 2(1.41\bar{6}) = 2.8\bar{3}$ $\quad \Delta d = \dfrac{2-(1.41\bar{6})^2}{2.8\bar{3}} = -0.00245$

$d_3 = 1.41\bar{6} - 0.00245 = 1.4142$ $\quad R_3 = |-6.007e{-6}| < 0.001\ (1.0)$ $\therefore$ CONVERGED !!

Rate of convergence: Provided certain conditions are satisfied, Newton-Raphson method exhibits quadratic convergence, i.e. $R_{i+1} \le C * (R_i)^2$

quadratic

e.g. $R_3 = 6.01e{-6} \le C * (6.94e{-3})^2$

number of zeros after decimal should double in succesive iterations!

Solving nonlinear problems poses a computational cost trade-off between rate of convergence (i.e. number of iterations) and tangent stiffness calculations (MDOF [S] inversion/factorization is expensive)

Modified Newton-Raphson scheme uses initial S_{tan} throughout load step, and will often converge to solution, but at a reduced rate ...

Back to example: $P^{ext} = d^2 = 0$, $\varepsilon = 0.001$

Standard Newton-Raphson (consistent tangent)

$d_0 = 1.0$, $S_t = 2.0$ $R_0 = 1.0$
$d_1 = 1.5$, $S_t = 3.0$ $R_1 = 0.25$
$d_2 = 1.41\overline{6}$, $S_t = 2.8\overline{3}$ $R_2 = 6.94e{-3}$
$d_3 = 1.4142 \longrightarrow$ $R_3 = 6.01e{-6}$ converged!

3 iterations

Modified Newton-Raphson (initial tangent)

$d_0 = 1.0$	$S_t = 2.0$	$R_0 = 1.0$
$d_1 = 1.50$	"	$R_1 = 0.25$
$d_2 = 1.375$	"	$R_2 = 0.1094$
$d_3 = 1.4297$	"	$R_3 = 0.0440$
$d_4 = 1.4077$	"	$R_4 = 0.0184$
$d_5 = 1.4169$	"	$R_5 = 0.0076$
$d_6 = 1.4131$	"	$R_6 = 0.0032$
$d_7 = 1.4147$	"	$R_7 = 0.0013$
$d_8 = 1.4140$	"	$R_8 = 5.41e{-4}$ converged!

8 iterations (less accurate)