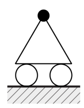
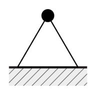
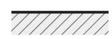
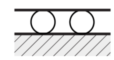
August 22, 2022
Matrix Displacement Method
Uniaxial Structures
Part I - Formulation

STRUCTURAL ANALYSIS PRELIMINARIES

Degrees of Freedom (DOFs) – independent joint displacements (translations and rotations) that are necessary to specify the deformed shape of the structure when subjected to an arbitrary loading.

(Q): How many DOFs for a 3D structure? 2D?

Supports, boundary conditions (BCs)

Roller  Pin  Fixed  Hybrid 

All correct solutions to mechanics problems must satisfy three principles:

1. ***Equilibrium*** – a structure initially at rest remains at rest when subjected to a system of forces and couples (i.e. moments).

   $$\sum F_x = F_y = F_z = M_x = M_y = M_z = 0 \qquad \text{(6 equations in 3D)}$$

2. ***Compatibility*** –structural elements do not overlap, have gaps, develop kinks or discontinuities; support conditions are satisfied.

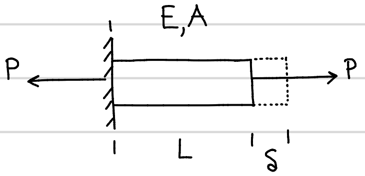
3. ***Constitutive relation*** – stress-strain behavior, provides link between equilibrium equations and compatibility conditions.

   $\sigma = E\varepsilon$ Hooke's law for linear, elastic material

   Statically determinate – equilibrium equations can be solved *independently* of constitutive relations to obtain reactions and member forces. Deformations can then be determined by employing compatibility and constitutive relations.

   Statically indeterminate – necessary to *simultaneously* solve three types of fundamental relationships (i.e. equilibrium, compatibility, and constitutive relation) in order to determine the structural response.

Uniaxial Structures



Equilibrium: $\sigma = P/A$ **Constitutive**: $\sigma = E\varepsilon$ **Kinematics**: $\varepsilon = \delta/L$

L = original length

E = elastic modulus

A = cross-sectional area

δ = displacement

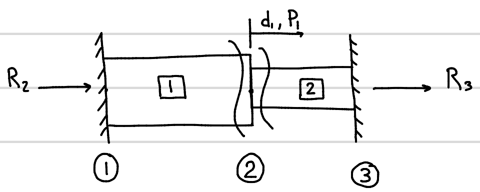
$$P/A = E\varepsilon = E\,\delta/L \qquad \delta = \frac{PL}{EA}$$

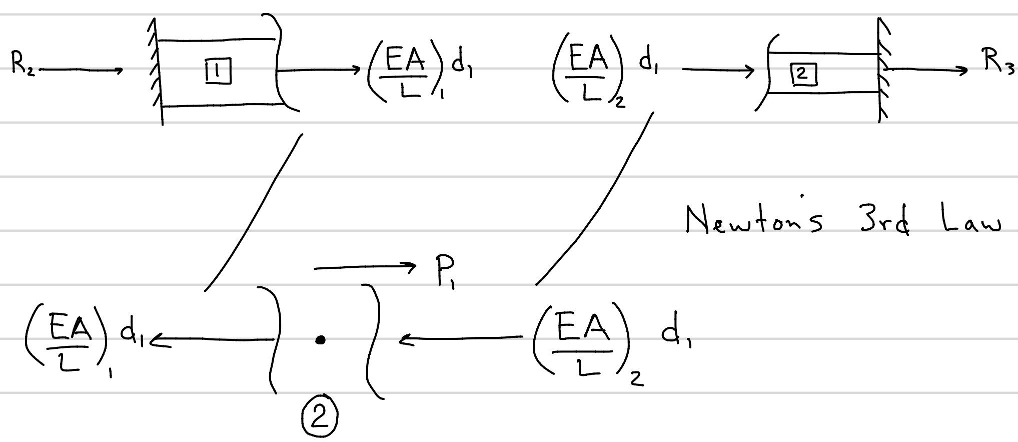
$$\underbrace{P}_{\text{force}} = \underbrace{\left(\frac{EA}{L}\right)}_{\text{stiffness}} \underbrace{\delta}_{\text{displacement}}$$

displacement → degree of freedom (DOF)

Spring: $F = kx$

1-DOF System





Newton's 3rd Law!

Notation

d = DOF [#] = member

P = force (#) = joint

R = reaction

Joint ② Equilibrium

$$\sum F_x = 0 \qquad P_1 - \left(\frac{EA}{L}\right)_1 d_1 - \left(\frac{EA}{L}\right)_2 d_1 = 0$$

$$P_1 = \left(\frac{EA}{L}_1 + \frac{EA}{L}_2\right) d_1$$

Structural-level relationship

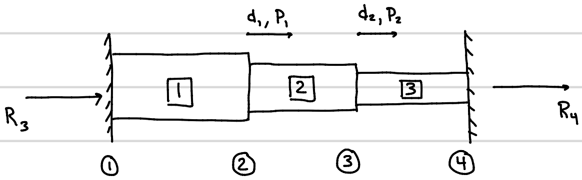
$$\underbrace{P_1}_{\text{externally applied load}} = S_{11}\, \underbrace{d_1}_{\text{DOF}}$$

S is structural stiffness comprised of member-level contributions

S_{ij} = stiffness at DOF i associated with a unit displacement at DOF j

Matrix Displacement Method (MDM)

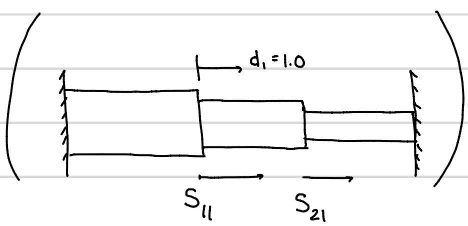
2-DOF



<u>Superposition</u>

* Structural Level *

=

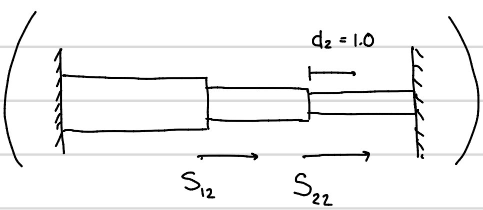


$* \; d_1$

$d_1 = 1 \quad d_2 = 0$

Note: Force at DOF i
= stiffness × displacement

+



$* \; d_2$

$d_1 = 0 \quad d_2 = 1$

Force Equilibrium

$$\{P\} = [S]\{d\}$$

Joint ② $\quad P_1 = S_{11} d_1 + S_{12} d_2$

Joint ③ $\quad P_2 = S_{21} d_1 + S_{22} d_2$

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$\{d\} = [S]^{-1} \{P\}$$

n # of dofs

$\{P\} = n \times 1$

$[S] = n \times n$

$$\underset{n\times 1}{\{P\}} = \underset{n\times n}{[S]}\underset{n\times 1}{\{d\}}$$

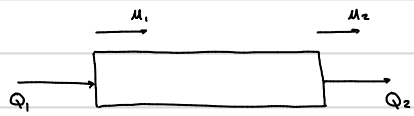
$[S]$ - symmetric, i.e. $S_{12} = S_{21}$

$\{d\} = n \times 1$

Member Level

$[S]$ is comprised of member stiffness contributions

generic uniaxial member 2 DOF



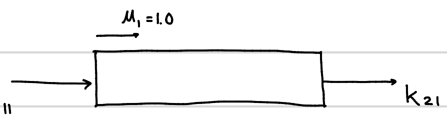
Q_i = force at DOF i

u_i = displacement at DOF i

k_{ij} = stiffness at DOF i associated w/ unit displacement at DOF j

=

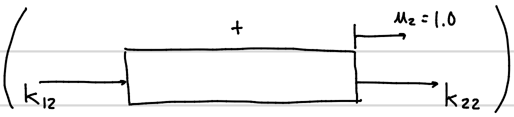
$u_1 = 1 \quad u_2 = 0$



$* \, u_1$

+

$u_1 = 0 \quad u_2 = 1$



$* \, u_2$

Force equilibrium at DOFs

$$Q_1 = k_{11} u_1 + k_{12} u_2$$

$$Q_2 = k_{21} u_1 + k_{22} u_2$$

$$\{Q\} = [k]\{u\}$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

what are the k_{ij} s ?