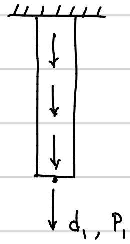
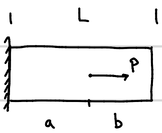
August 29, 2024
Matrix Displacement Method
Uniaxial Structures
Part IV - Fixed End Forces
Non-prismatic Bar



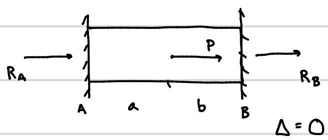
How to handle loads NOT applied at DOFs ? e.g. self-weight

Matrix Displacement Method (MDM) "discretizes" structures into elements with loads applied at degrees of freedom (nodes).

Equivalent nodal loading is determined from technique known by Fixed-end-forces (FEF).



E,A



Fixed End Forces

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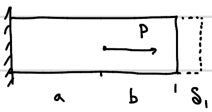
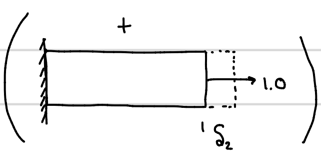
* Use force superposition *



Remove redundant reaction (apply loading)

\+



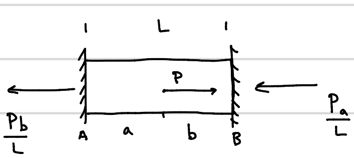
$* R_B$ Apply scaled unit reaction

Displacement compatibility : $\Delta = \delta_1 + \delta_2 = 0$ $\qquad \delta = \dfrac{PL}{EA} \qquad \delta_1 = \dfrac{Pa}{EA} \qquad \delta_2 = \left(\dfrac{1.0\,L}{EA}\right) * R_B$

$$\frac{Pa}{EA} + \frac{R_B L}{EA} = 0 \qquad \therefore \quad R_B = -\frac{Pa}{L}$$

Using overall force equilibrium : $\Sigma F_x = 0 \qquad R_A + R_B + P = 0 \qquad R_A - \dfrac{Pa}{L} + P = 0$

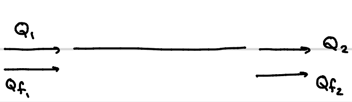
$$R_A = \frac{Pa}{L} - P = \frac{Pa - PL}{L} = \frac{-P(L-a)}{L} = -\frac{Pb}{L}$$



* Member level *

$\{Q\} = [k]\{u\}$ w/o interior loads

$\{Q\} = \{Q_f\} + [k]\{u\}$ w/ interior loading



$\{Q - Q_f\} = [k]\{u\}$

member-end forces

fixed end forces from interior loads

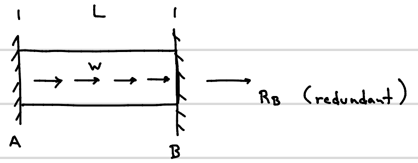
(negative sign = opposite direction of reactions)

* Structural Level *

$\{P - P_f\} = [S]\{d\}$

force vector for external loads applied at DOFs

assemble fixed-end-force vector from member-level Qf contributions



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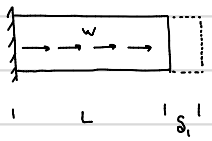
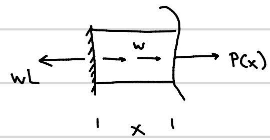
$\Delta = 0$



$$\delta(x) = \int \frac{P(x)\,dx}{EA}$$



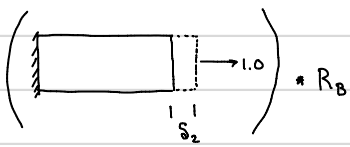
$\Sigma F_x = 0 \qquad P(x) + wx - wL = 0$

$P(x) = wL - wx$

$$\delta(x) = \int \frac{w(L-x)}{EA}\,dx = \frac{w}{EA}\int_0^L (L-x)\,dx = \frac{w}{EA}\Big|_0^L Lx - \frac{x^2}{2} = \frac{wL^2}{2EA}$$

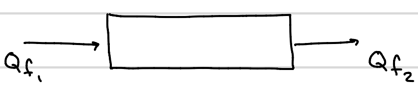
δ_1

+



$\Delta = \delta_1 + \delta_2 = 0 \qquad \delta_2 = \dfrac{R_B L}{EA} \qquad \dfrac{wL^2}{2EA} + \dfrac{R_B L}{EA} = 0$

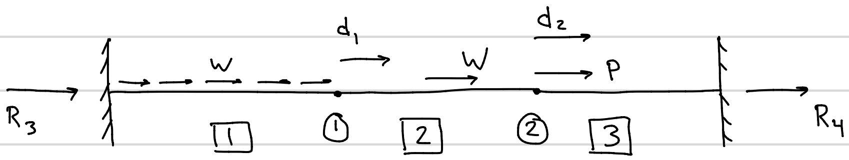
$R_B = \dfrac{-wL}{2} \quad \therefore \quad R_A = \dfrac{-wL}{2}$



$$\{Q_f\} = \begin{Bmatrix} R_A \\ R_B \end{Bmatrix} = \begin{Bmatrix} -wL/2 \\ -wL/2 \end{Bmatrix}$$

member-level $\quad \{Q - Q_f\} = [k]\{u\}$

structural-level $\quad \{P - P_f\} = [S]\{d\}$



$\{P - P_f\} = [S]\{d\}$

↓

$\begin{Bmatrix} 0 \\ P \end{Bmatrix}$ force vector applied at DOFs

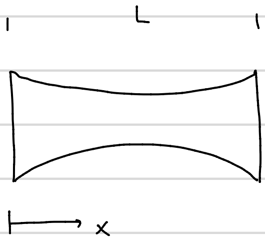
	u_1, Q_1	u_2, Q_2
$\boxed{1}$	3	1
$\boxed{2}$	1	2
$\boxed{3}$	2	4

$$[S] = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & \quad 2 \end{array} \\ \begin{bmatrix} k_{22}^1 \ \cdots & \cdots \\ \cdots & \cdots \end{bmatrix} \end{array}$$

$$\{P_f\} = \begin{array}{c} 1 \\ 2 \end{array}\begin{Bmatrix} Q_{f_2}^1 + Q_{f_1}^2 \\ Q_{f_2}^2 \end{Bmatrix}$$

$$[k]^1 = \begin{array}{c} \\ 3 \\ 1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 3 & \quad 1 \end{array} \\ \begin{bmatrix} k_{11}^1 & k_{12}^1 \\ k_{21}^1 & k_{22}^1 \end{bmatrix} \end{array} \qquad \{Q_f\}^1 = \begin{array}{c} 3 \\ 1 \end{array}\begin{Bmatrix} Q_{f_1}^1 \\ Q_{f_2}^1 \end{Bmatrix} \qquad \{Q_f\}^2 = \begin{array}{c} 1 \\ 2 \end{array}\begin{Bmatrix} Q_{f_1}^2 \\ Q_{f_2}^2 \end{Bmatrix}$$

Non-prismatic Bar



$A(x)$

E

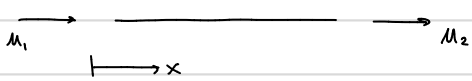
$\sigma = E\varepsilon \qquad \sigma = \frac{P(x)}{A(x)} \qquad \varepsilon = \frac{d\bar{u}_x}{dx}$ $\quad \bar{u}_x$ = axial deformation

$$\frac{d\bar{u}_x}{dx} = \frac{P(x)}{EA(x)} \qquad \bar{u}_x = \int \frac{P(x)}{EA(x)}\,dx$$

homogeneous

linear elastic (Hooke's Law)

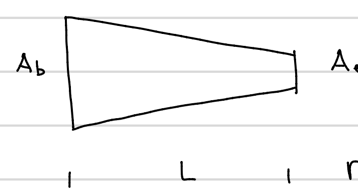
2 DOF uniaxial element



1st column of [k]

$\Sigma F_x = 0 \qquad P(x) = -k_{11} * 1.0$ (stiffness x displacement)

$$\bar{u}_x = \int \frac{P(x)}{EA(x)}\,dx = \frac{1}{E}\int \frac{-k_{11}}{A(x)}\,dx$$



$$r_A = \frac{A_b - A_e}{A_b}$$

$$\bar{u}_x = \frac{1}{E}\int \frac{-k_{11}}{A_b\left(1 - r_A\frac{x}{L}\right)}\,dx$$

$$A(x) = A_b\left(1 - \frac{r_A x}{L}\right)$$

linearly varying area (tapered bar)

$$* \quad \bar{u}_x = \frac{-k_{11}}{EA_b}\ln\left(1 - \frac{r_A x}{L}\right)\frac{-L}{r_A} + C$$

B.C.s at $x = 0 \quad \bar{u}_x = u_1 = 1.0$

$x = L \quad \bar{u}_x = u_2 = 0$

from 1st B.C. $\quad C = 1.0 \quad\quad {}^{*}\ln(1) = 0$

2nd B.C. $\quad 0 = \dfrac{k_{11} L}{E A_b r_a} \ln(1 - r_a) + 1.0$

solve for ${}^{*}k_{11} = -\dfrac{E A_b r_a}{L \ln(1 - r_a)}$

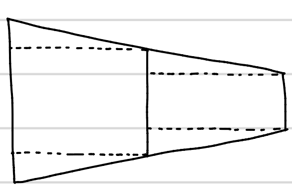
from equilibrium $\Sigma F_x = 0 \quad k_{11} + k_{21} = 0 \quad \therefore \quad k_{21} = \dfrac{E A_b r_a}{L \ln(1 - r_a)}$

similarly column 2 of the stiffness matrix $(u_2 = 1.0,\ u_1 = 0)$ can be derived to obtain

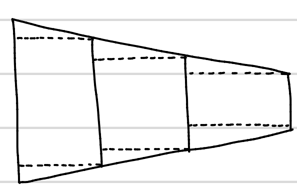
$$[k] = \frac{E A_b r_a}{L \ln(1 - r_a)} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Exact stiffness matrix for linearly tapered (area) bar element

where $r_a = \dfrac{A_b - A_e}{A_b}$



Approximate solutions can be obtained by using multiple prismatic bar elements



Closer approximations by subdividing into more elements (h-refinement)

Finite-Element Concept !