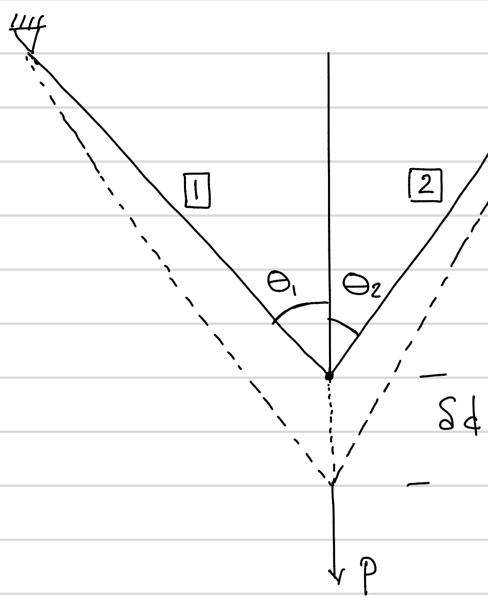
September 03, 2024
Principle of Virtual Work
Part I - Formulation
Uniaxial Structures

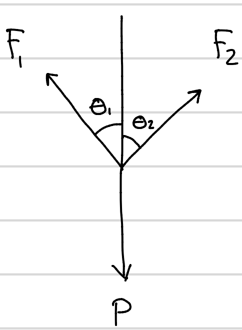
Principle of Virtual Work (P.V.W.)

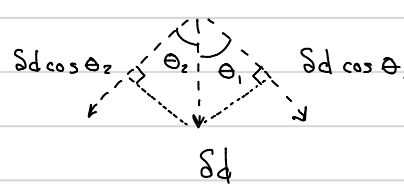
(for deformable bodies)

If a deformable structure, which is in equilibrium under a system of forces (and couples) is subjected to any small virtual displacement (δd) consistent with the support and continuity conditions of the structure, then the virtual external work done by real external forces (and couples) acting through the virtual external displacements (and rotations) is equal to the virtual strain energy (internal work) stored in the structure.

External Virtual Work = Internal Virtual Work







Real Joint Forces

Virtual Displacements

Equilibrium

$\sum F_x = 0 \qquad -F_1 \sin\theta_1 + F_2 \sin\theta_2 = 0$

$^* \sum F_y = 0 \qquad F_1 \cos\theta_1 + F_2 \cos\theta_2 - P = 0$

Virtual Work
(δW)

$W = F \cdot d = |F||d| \cos\phi$
(dot product)

$\xrightarrow{F} \xrightarrow{\delta d} \qquad F\delta d \cos(0) = F\delta d$

$\xleftarrow{F} \xrightarrow{\delta d} \qquad F\delta d \cos(180) = -F\delta d$

$$\delta W = P\delta d - F_1(\delta d \cos\theta_1) - F_2(\delta d \cos\theta_2)$$

$$= \delta d \underbrace{(P - F_1\cos\theta_1 - F_2\cos\theta_2)}_{^* \sum F_y = 0}$$

$\therefore \quad \delta W = 0$

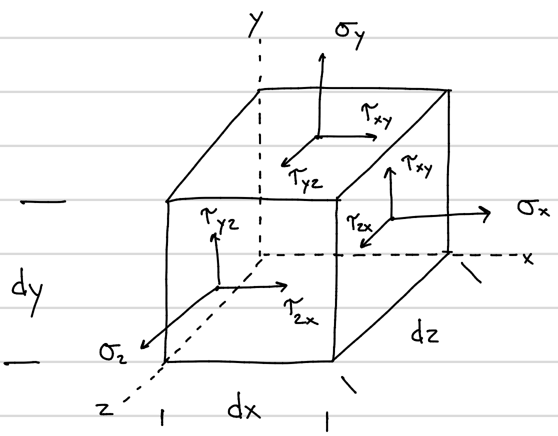
$$P\delta d = F_1(\delta d \cos\theta_1) + F_2(\delta d \cos\theta_2)$$

External Virtual Work (δW_e)

Internal Virtual Work (δU) internal stored energy

$$\boxed{\delta W_e = \delta U}$$

δU internal virtual work, i.e. strain energy



real internal force $= \sigma \cdot A = \sigma_x \, (dy\, dz)$

virtual displacement $= \delta\varepsilon \cdot L = \delta\varepsilon_x \, (dx)$

$\sigma = F/A$

$\varepsilon = \Delta/L$

virtual internal work $= (\sigma_x \, dy\, dz)(\delta\varepsilon_x \, dx)$

$$= \delta\varepsilon_x \, \sigma_x \, \underbrace{dx\, dy\, dz}$$

$$= \delta\varepsilon_x \, \sigma_x \, dV$$

$$d\delta U = (\delta\varepsilon_x \sigma_x + \delta\varepsilon_y \sigma_y + \delta\varepsilon_z \sigma_z$$
$$\delta\gamma_{xy} \tau_{xy} + \delta\gamma_{yz} \tau_{yz} + \delta\gamma_{zx} \tau_{zx}) \, dV$$

$$\delta U = \int_V \delta\varepsilon^T \sigma \, dV \qquad \delta\varepsilon = \begin{Bmatrix} \delta\varepsilon_x \\ \delta\varepsilon_y \\ \delta\varepsilon_z \\ \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix} \qquad \sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

$$\boxed{\delta W_e = \delta U = \int_V \delta\varepsilon^T \sigma \, dV}$$

P.V.W.

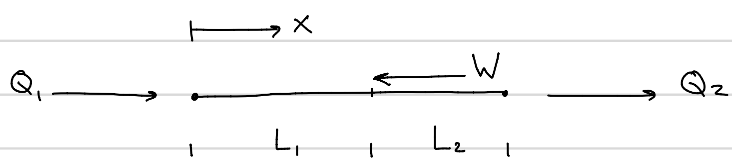
Uniaxial P.V.W.

$$\delta W_{ext} = \delta W_{int}$$

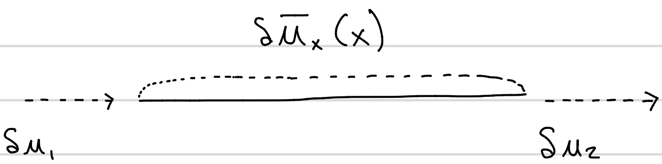
real force × virtual displacement

strain energy δU

real forces



virtual displacements



$$\delta W_{ext} = Q_1 \delta u_1 + Q_2 \delta u_2 - W \delta \bar{u}_x(L_1)$$

$$= \delta u^T Q - W \delta \bar{u}_x(L_1)$$

$$\{\delta u\}^T = \{\delta u_1 \;\; \delta u_2\}$$

What is $\delta \bar{u}_x(x)$??

$$\{Q\} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

Finite Element Method (FEM) - assumes a displacement function (complete polynomials)

$$\bar{u}_x(x) = \sum_{i=0}^{n} a_i x^i, \; a_i \neq 0$$

n = B.C.s − 1

Uniaxial B.C.s = 2

$n = 1$ linear

$$\bar{u}_x = a_0 x^0 + a_1 x^1$$
$$= a_0 + a_1 x$$

1

$x \quad y$

$x^2 \quad xy \quad y^2$

$x^3 \quad x^2y \quad xy^2 \quad y^3$