September 05, 2024
Principle of Virtual Work
Part II - Finite Element Method
Uniaxial Structures

$\bar{u}_x = a_0 + a_1 x$ B.C.s @ $x = 0$ $\bar{u}_x = u_1$

@ $x = L$ $\bar{u}_x = u_2$

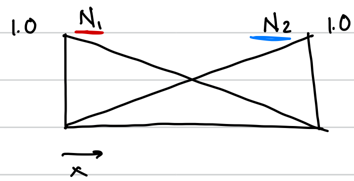
$\therefore \quad a_0 = u_1$

$$a_1 = \frac{u_2 - u_1}{L}$$

$$\bar{u}_x = u_1 + \left(\frac{u_2 - u_1}{L}\right) x \qquad \bar{u}_x = \underbrace{\left(1 - \frac{x}{L}\right)}_{N_1} u_1 + \underbrace{\left(\frac{x}{L}\right)}_{N_2} u_2$$

shape functions N_i

$$\bar{u}_x = N_1 u_1 + N_2 u_2 = \underline{\mathbf{N} u}$$



$$\mathbf{N} = \{ N_1 \quad N_2 \} \qquad u = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\delta \bar{u}_x = \mathbf{N} \, \underline{\delta u}$$ (Bubnov - Galerkin)

$$* \quad \delta W_{int} = \int_V \underline{\delta \varepsilon^T} \, \underline{\sigma} \, dV$$

$\varepsilon = \dfrac{d \bar{u}_x}{dx}$ $\bar{u}_x = \mathbf{N} u$ strain - displacement (kinematics)

$$\varepsilon = \frac{d(\mathbf{N} u)}{dx} = \frac{d\mathbf{N}}{dx} u + \cancelto{\emptyset}{\frac{du}{dx} \mathbf{N}} = \underbrace{\frac{d\mathbf{N}}{dx}}_{\mathbf{B}} u$$

$\mathbf{N} = \{ N_1 \quad N_2 \} = \left\{ 1 - \frac{x}{L} \quad \frac{x}{L} \right\}$ linear displacement

$\mathbf{B} = \{ B_1 \quad B_1 \} = \left\{ -\frac{1}{L} \quad \frac{1}{L} \right\}$ constant strain

$$\delta W_{int} = \int_V \delta \varepsilon^T \sigma \, dV$$

$\varepsilon = \mathbf{B} u$ $\qquad \sigma = E \varepsilon = E \mathbf{B} u$ (constitutive relation)

$\delta \varepsilon = \mathbf{B} \delta u$ $\qquad\qquad\qquad\qquad$ linear elastic

$$\delta W_{int} = \int_V \overset{*}{(}\mathbf{B} \delta u)^T E \mathbf{B} u \, dV$$

$$= \int_V \underset{\text{constant}}{\delta u^T} \, \underset{\text{constant}}{\mathbf{B}^T E} \, \underset{\text{constant}}{\mathbf{B} u} \, dV$$

$$= \delta u^T \overset{*}{\left(} \int_V \mathbf{B}^T E \mathbf{B} \, dV \overset{*}{\right)} u$$

<u>P.V.W.</u> $\quad \delta W_{ext} = \underline{\delta W_{int}}$ $\qquad\qquad \delta W_{ext} - \delta W_{int} = 0$

$$\delta W_{ext} = \underbrace{Q_1 \delta u_1 + Q_2 \delta u_2}_{} - W \delta \bar{u}_x (L_1)$$

$$= \delta u^T \mathbf{Q} - \underline{W \delta \bar{u}_x (L_1)}$$

$$\underline{- W \delta \bar{u}_x (L_1)} = - W \underset{1\times2}{\mathbf{N}(L_1)} \, \underset{2\times1}{\delta u}$$

$$= - W \underset{1\times2}{\delta u^T} \, \underset{2\times1}{N^T (L_1)} \; *$$

$$\underset{1\times2}{\{N_1 \;\; N_2\}} \qquad \underset{2\times1}{\begin{Bmatrix} \delta u_1 \\ \delta u_2 \end{Bmatrix}}$$

$$\{\delta u_1 \;\; \delta u_2\} \qquad \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix}$$

$$\delta u^T Q - \overset{*}{W} \delta u^T N^T (L_1\overset{*}{)} - \delta u^T \left(\int_V B^T E B \, dV \right) u = 0$$

$$\underbrace{\delta u^T Q - \delta u^T W N^T (L_1)}_{\delta W_{ext}} - \underbrace{\delta u^T \left(\int_V B^T E B \, dV \right) u}_{\delta W_{int}} = 0$$

$$\delta u^T \left(Q - W N^T (L_1) - \int_V B^T E B dV \, u \right) = 0$$

δu^T arbitrarily chosen and not $= 0$ $\;\therefore\;$ $(\qquad) = 0$

$$* \quad \underline{Q} - \underline{W N^T(L_1)} - \underline{\int_V B^T E B \, dV \, \underline{u}} = 0$$

Recall $\{Q\} = \{Qf\} + [k]\{u\}$ member-level equilibrium equation

$$* \quad \underline{\{Q\}} - \underline{\{Qf\}} - \underline{[k]}\{u\} = 0$$

$$\{Qf\} \rightarrow W N^T(L_1) \qquad [k] \rightarrow \int_V B^T E B \, dV$$

$$[k] = \int_V B^T E B \, dV \qquad B = \frac{dN}{dx} = \frac{d}{dx}\left\{1 - \frac{x}{L} \quad \frac{x}{L}\right\} = \underset{1\times 2}{\left\{-\frac{1}{L} \quad \frac{1}{L}\right\}}$$

$dV \rightarrow \underline{dA}\,dx \rightarrow$ if A is constant, i.e. prismatic

E is also constant

$$= EA \int_0^L B^T B \, dx$$

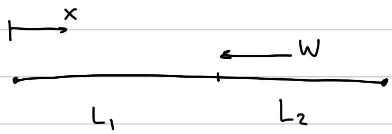
$$\underset{2\times 1}{\begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix}} \underset{1\times 2}{\left\{-\frac{1}{L} \quad \frac{1}{L}\right\}} = \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix}$$

$$= EA \int_0^L \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} dx$$

$$= \frac{EA}{L^2} \int_0^L \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx = \frac{EA}{L^2} \Big|_0^L \begin{bmatrix} x & -x \\ -x & x \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

equivalent to direct method

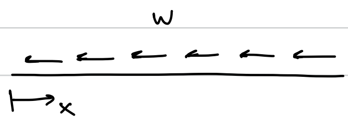
What about Qf ?

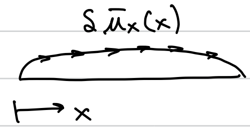


$^*\; L - L_1 = L_2 \;^*$

$$Qf = W\, N^T(L_1)$$

$$= W \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix}_{x=L_1} = W \begin{Bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{Bmatrix}_{x=L_1} = W \begin{Bmatrix} 1 - \frac{L_1}{L} \\ \frac{L_1}{L} \end{Bmatrix} = W \begin{Bmatrix} \frac{L-L_1}{L} \\ \frac{L_1}{L} \end{Bmatrix} = W \begin{Bmatrix} \frac{L_2}{L} \\ \frac{L_1}{L} \end{Bmatrix}$$





$$\delta W_{ext} = - \int w(x)\, \underset{\substack{\downarrow \\ N\, \delta u \\ \delta u^T N^T}}{\delta \bar{u}_x(x)}\, dx = - \delta u^T \underbrace{\int w(x)\, N^T(x)\, dx}_{^*\{Qf\}^*}$$

$$\{Qf\} = \int w(x)\, N^T(x)\, dx \quad \text{uniform load } w(x) = w \text{ i.e. constant}$$

$$w \int N^T(x)\, dx = w \int_0^L \begin{Bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{Bmatrix} dx = w \Big|_0^L \begin{Bmatrix} x - \frac{x^2}{2L} \\ \frac{x^2}{2L} \end{Bmatrix} = \underbrace{w \begin{Bmatrix} \frac{L}{2} \\ \frac{L}{2} \end{Bmatrix}}_{Qf}$$