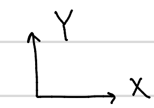
September 10, 2024
Matrix Displacement Method
2D Trusses
Part I - Local/Global

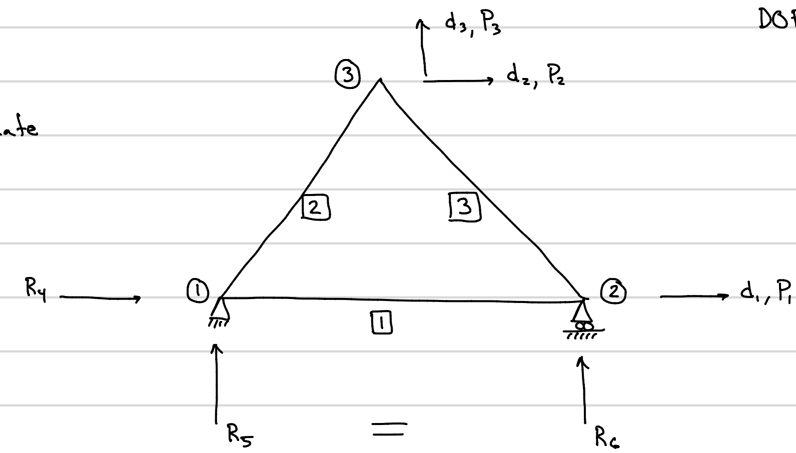
Trusses (2D planar)

Criteria :

1) consist of weightless uniaxial elements with arbitrary angular orientation and interconnected with frictionless pins (No moment)
2) subjected to loads only at joints (No P_f)

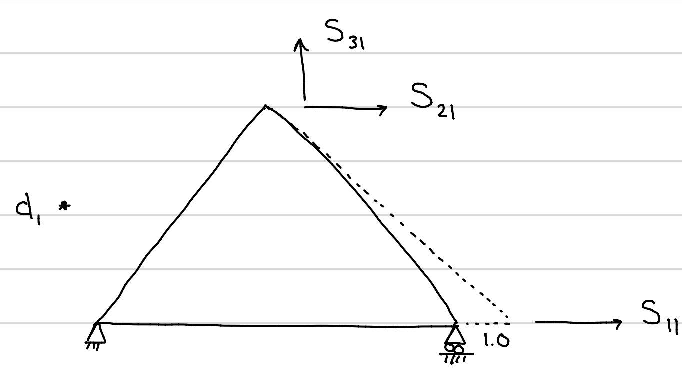


global coordinate system



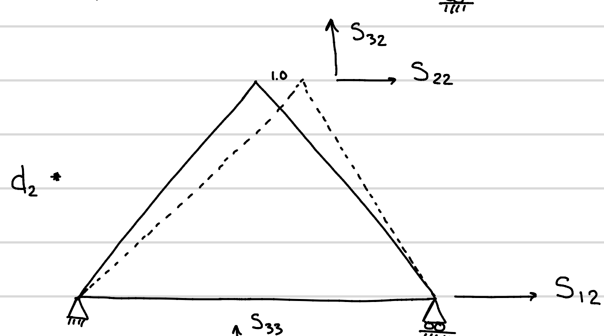
DOFs truss : 2 ⟶ X and Y translations

* Superposition *



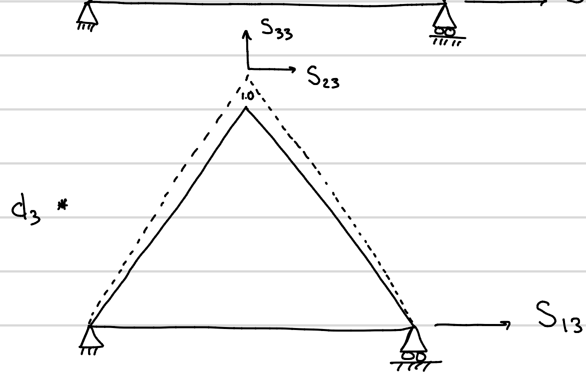
$d_1 = 1.0$, $d_2 = d_3 = 0$

+



$d_2 = 1.0$, $d_1 = d_3 = 0$

+



$d_3 = 1.0$, $d_1 = d_2 = 0$

From superposition

$$\{P\} = [S]\{d\}$$

$$P_1 = S_{11} d_1 + S_{12} d_2 + S_{13} d_3$$

$$P_2 = S_{21} d_1 + S_{22} d_2 + S_{23} d_3$$

$$P_3 = S_{31} d_1 + S_{32} d_2 + S_{33} d_3$$

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

structural-level

* Difference in $[S]$ comes from member-level *

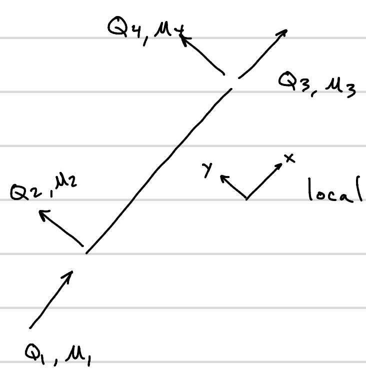
Notation

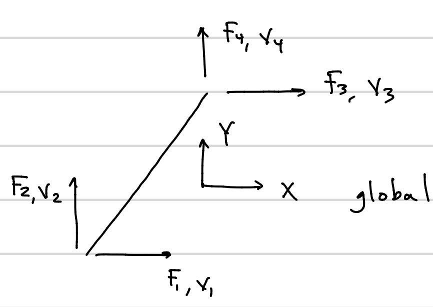
$\{P\} = [S]\{d\}$ — structural-level (always global)

force — stiffness — displacement

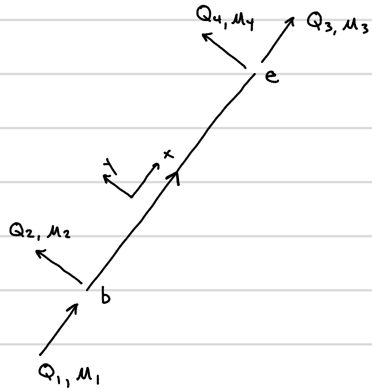
x, y: $\{Q\} = [k]\{u\}$ — member-level (local)

X, Y: $\{F\} = [K]\{v\}$ — member-level (global)



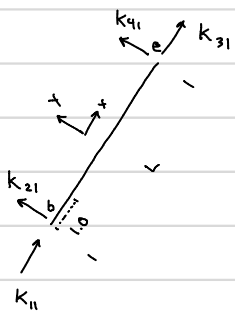


$\{Q\} = [k]\{u\}$ member-level (local)



How do we determine $[k]$

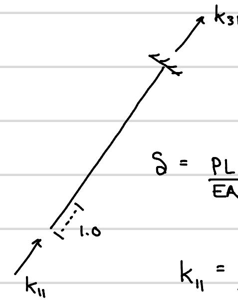
$u_1 = 1$, $u_2 = u_3 = u_4 = 0$ (1st column)



$\Sigma F_x = 0$ $\quad k_{11} + k_{31} = 0$ $\quad k_{11} = -k_{31}$

$\Sigma F_y = 0$ $\quad k_{21} + k_{41} = 0$ $\quad \therefore k_{41} = 0$

$\Sigma M_e = 0$ $\quad -k_{21} * L = 0$ $\quad k_{21} = 0$

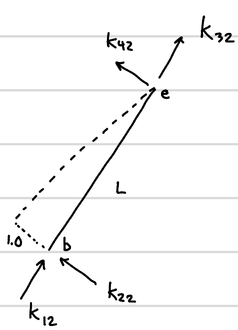


$\delta = \frac{PL}{EA}$ $\quad P = \frac{EA}{L}\delta$

$k_{11} = \frac{EA}{L}(1.0) \therefore$

$k_{11} = \frac{EA}{L}$ $\quad k_{31} = \frac{-EA}{L}$

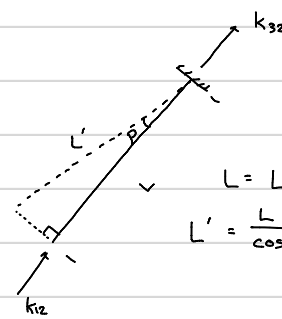
$u_2 = 1$, $u_1 = u_3 = u_4 = 0$ (2nd column)



$\Sigma F_x = 0$ $\quad k_{12} + k_{32} = 0$ $\quad k_{12} = -k_{32}$

$\Sigma F_y = 0$ $\quad k_{22} + k_{42} = 0$ $\quad \therefore k_{42} = 0$

$\Sigma M_e = 0$ $\quad -k_{22} * L = 0$ $\quad k_{22} = 0$



$L = L' \cos p$

$L' = \frac{L}{\cos p}$

small rotation ($p << 1$)

$\therefore \cos p \approx 1$

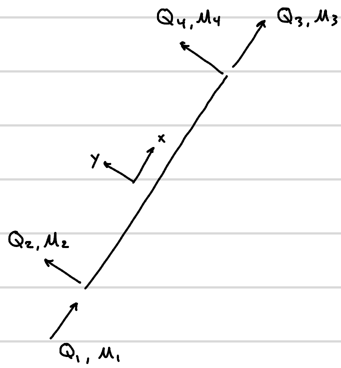
$L' \approx L$ $\quad \underbrace{L' - L}_{\delta_{axial}} \approx 0$

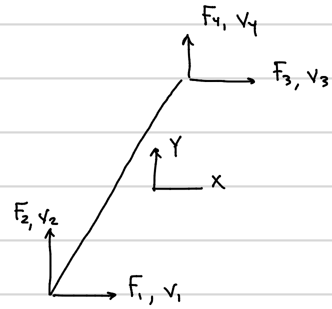
$\therefore$ No (i.e. negligible) axial force develops from transverse displacement $k_{12} = -k_{32} = 0$

Using same procedure for u_3 and u_4 we arrive upon $[k]$

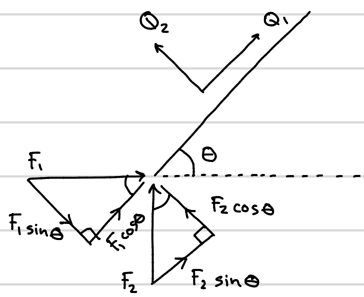
$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Local vs. Global





We need to write Qs in terms of Fs



$$Q_1 = F_1 \cos\theta + F_2 \sin\theta$$

$$Q_2 = -F_1 \sin\theta + F_2 \cos\theta$$

* similarly for Q_3, Q_4

$$\underset{4\times1}{\{Q\}} = \underset{4\times4}{[T]} \underset{4\times1}{\{F\}}$$

↑ transformation matrix

$$[T] = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \qquad c \equiv \cos\phi \quad s \equiv \sin\phi$$

forces $\{Q\} = [T]\{F\}$

displacements $\{u\} = [T]\{v\}$

$$\{Q\} = [k]\{u\} \qquad \underbrace{[T]\{F\}}_{\{Q\}} = [k]\underbrace{[T]\{v\}}_{\{u\}}$$

$$\underbrace{[T]^{-1}[T]}_{[I]}\{F\} = [T]^{-1}[k][T]\{v\}$$

[T] is orthogonal

$$[T]^{-1} = [T]^T$$

$$\{F\} = \underbrace{[T]^T[k][T]}_{[K]}\{v\}$$

[K] member-level global stiffness matrix

$$[K] = \frac{EA}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$