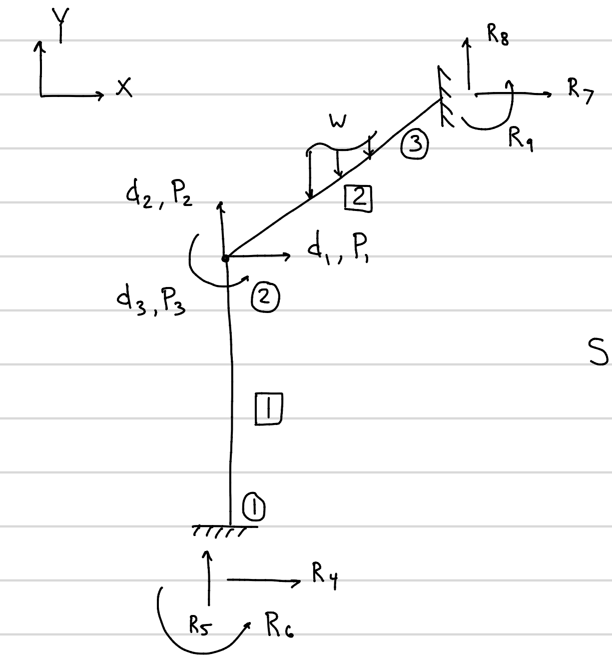
April 07, 2026
Matrix Displacement Method
2D Frames
Part I - formulation, [k], [T]

2D Frame Analysis (MDM)

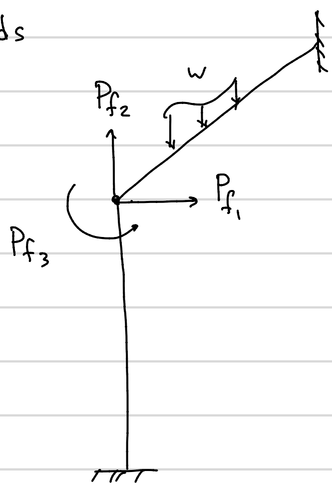


3 DOFs - X, Y translations, rotation Z

Structural-Level Superposition

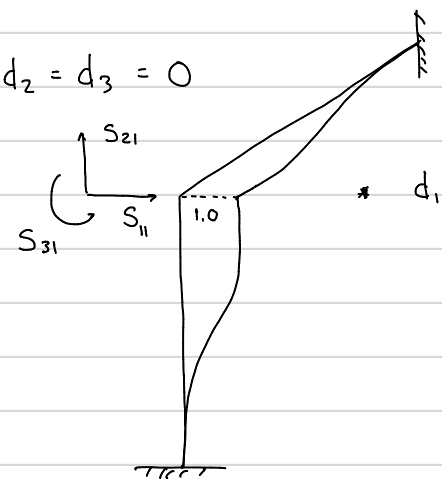
=

Apply member loads
$d_1 = d_2 = d_3 = 0$



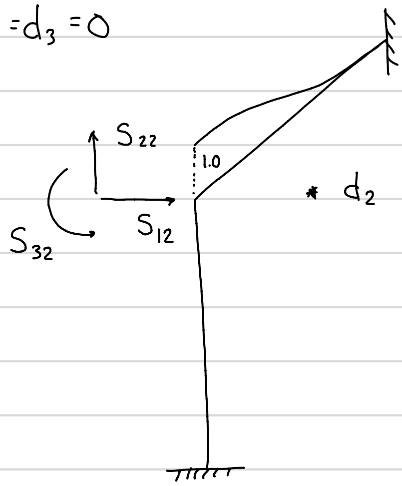
+

$d_1 = 1$, $d_2 = d_3 = 0$



$* \; d_1$

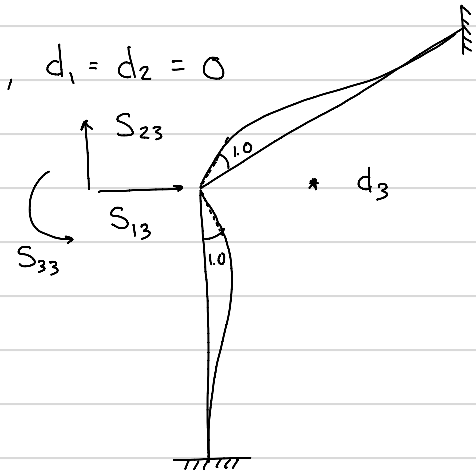
$d_2 = 1$, $d_1 = d_3 = 0$



$* \; d_2$

+

$d_3 = 1$, $d_1 = d_2 = 0$



$* \; d_3$

$$P_1 = P_{f_1} + S_{11} d_1 + S_{12} d_2 + S_{13} d_3$$

Joint equilibrium at DOFs

$$P_2 = P_{f_2} + S_{21} d_1 + S_{22} d_2 + S_{23} d_3$$

$$P_3 = P_{f_3} + S_{31} d_1 + S_{32} d_2 + S_{33} d_3$$

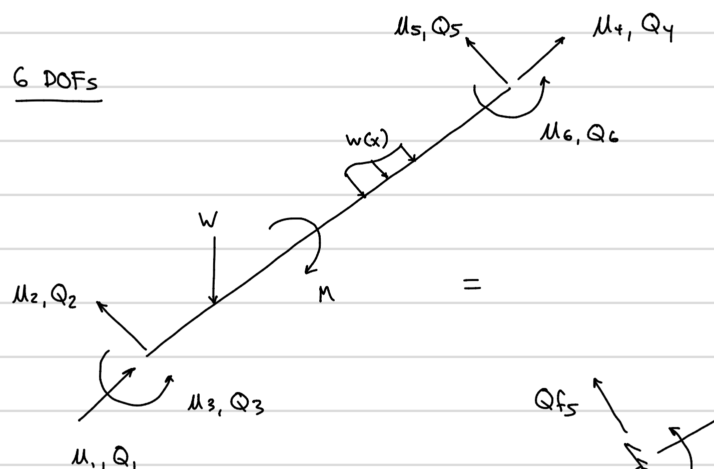
$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} P_{f_1} \\ P_{f_2} \\ P_{f_3} \end{Bmatrix} + \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

$$\{P\} = \{P_f\} + [S]\{d\}$$

Structural-level
linear system of equations

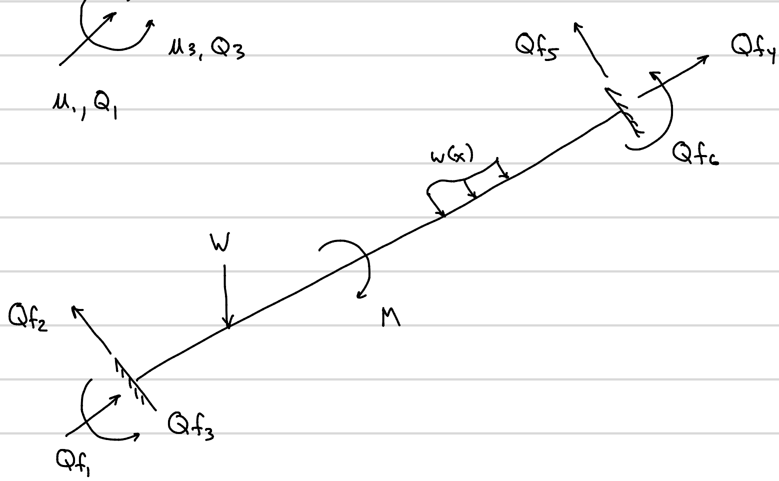
Member-level Superposition

<u>6 DOFs</u>

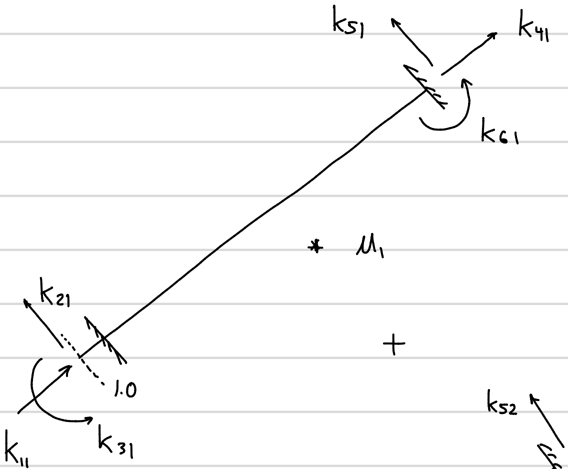


local coordinates to develop
stiffness matrix $[k]$

=



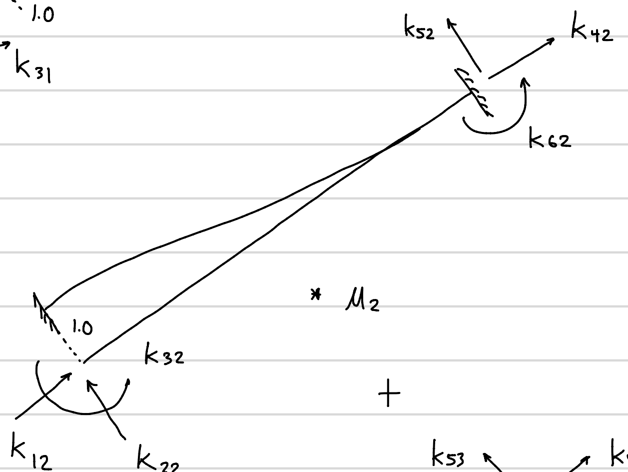
Apply member (interior) loads
$u_1 \ldots u_6 = 0$



$u_1 = 1 \ , \ u_2 \dots u_6 = 0$

$* \ u_1$

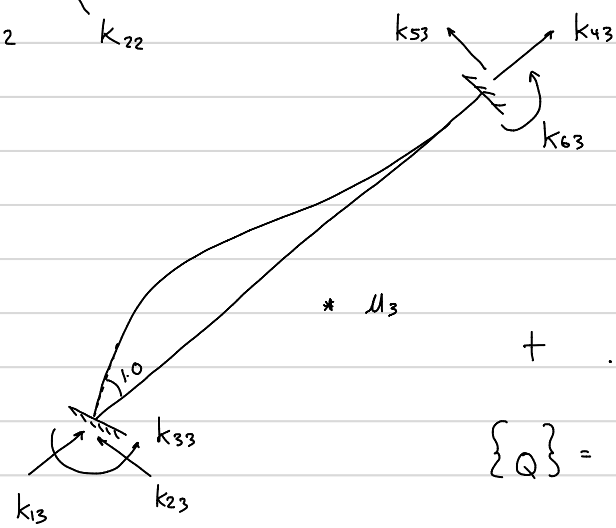
$+$



$u_2 = 1 \ , \ u_1 \ , \ u_3 \dots u_6 = 0$

$* \ u_2$

$+$



$u_3 = 1.0 \ , \ u_1 \ , u_2 \ , u_4 \dots u_6 = 0$

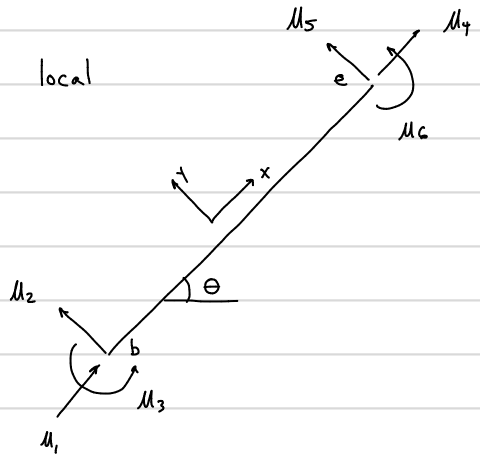
$* \ u_3$

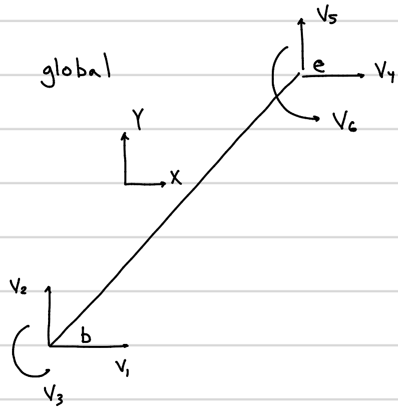
$+$ .... similarly for $u_4 - u_6 = 1.0$

$$\{Q\} = \{Q_f\} + [k]\{u\}$$

Using same analysis procedures to derive uniaxial/beam stiffness terms, $[k]$ for 2D frame:

$$[k]_{6\times6} = \begin{bmatrix} EA/L & 0 & 0 & -EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -EA/L & 0 & 0 & EA/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix}$$





$$u_1 = v_1 \cos\theta + v_2 \sin\theta$$

$$u_2 = -v_1 \sin\theta + v_2 \cos\theta$$

$$u_3 = v_3$$

$$\underset{6\times1}{\{u\}} = \underset{6\times6}{[T]}\,\underset{6\times1}{\{v\}}$$

$$\underset{6\times1}{\{Q\}} = \underset{6\times6}{[T]}\,\underset{6\times1}{\{F\}}$$

$$[T] = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\{F\} = [T]^T\{Q\}$$

$$\{Q\} = \{Q_f\} + [k]\{u\}$$

$$\{u\} = [T]\{v\}$$

$$\{F\} = \underbrace{[T]^T\{Q_f\}}_{\{F_f\}} + \underbrace{[T]^T[k][T]}_{[K]}\{v\}$$

Global member-end forces

$$\therefore \quad \{F\} = \{F_f\} + [K]\{v\}$$