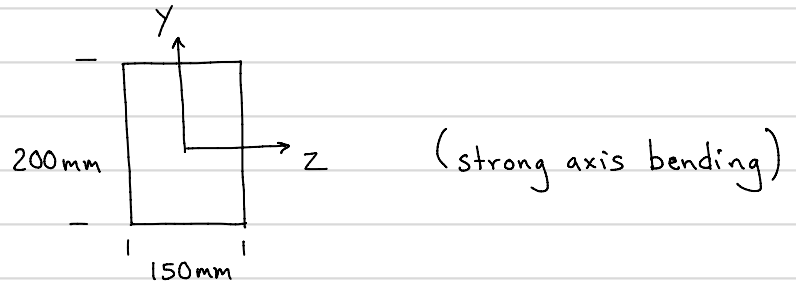
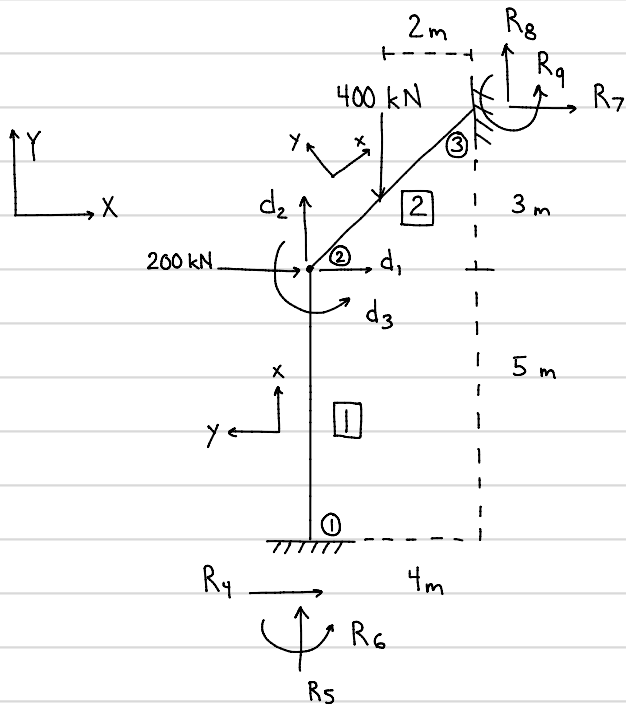


2D Frame Example

1. Label Structure

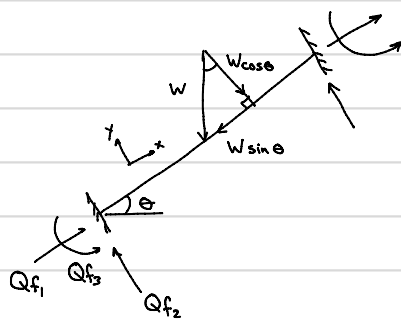


$E = 20 \text{ GPa}$ for all members

$$2. \quad \begin{Bmatrix} P \\ \end{Bmatrix}_{3 \times 1} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} 200 \\ 0 \\ 0 \end{Bmatrix} \text{ kN}$$

3. Member-level contributions (local)

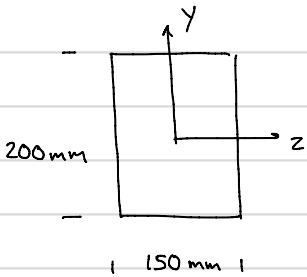
$$\{Q_f\}' = \{\emptyset\}$$



$$\{Q_f\}^2 = \begin{Bmatrix} F_{Ab} \\ F_{Sb} \\ F_{Mb} \\ F_{Ae} \\ F_{Se} \\ F_{Me} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} W \sin \theta \\ \frac{1}{2} W \cos \theta \\ \frac{1}{8} WL \cos \theta \\ \frac{1}{2} W \sin \theta \\ \frac{1}{2} W \cos \theta \\ -\frac{1}{8} WL \cos \theta \end{Bmatrix} = \begin{Bmatrix} 120 \\ 160 \\ 2e5 \\ 120 \\ 160 \\ -2e5 \end{Bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN} \cdot \text{mm} \\ \text{kN} \\ \text{kN} \\ \text{kN} \cdot \text{mm} \end{matrix}$$

$$[K] = \begin{bmatrix} EA/L & 0 & 0 & -EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -EA/L & 0 & 0 & EA/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix}$$

$$E = 20 \text{ GPa} \quad (GPa = \text{kN/mm}^2)$$



$$A = (150)(200) = 3e4 \text{ mm}^2$$

$$I_z = \frac{(150)(200)^3}{12} = 1e8 \text{ mm}^4$$

$$L^1 = L^2 = 5,000 \text{ mm}$$

$$[k]_{1,2} =$$

$$\begin{bmatrix} 120 & 0 & 0 & -120 & 0 & 0 \\ 0 & 0.192 & 480 & 0 & -0.192 & 480 \\ 0 & 480 & 1.6e6 & 0 & -480 & 0.8e6 \\ -120 & 0 & 0 & 120 & 0 & 0 \\ 0 & -0.192 & -480 & 0 & 0.192 & -480 \\ 0 & 480 & 0.8e6 & 0 & -480 & 1.6e6 \end{bmatrix}$$

transformation

$$[T] = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

1

$$c = 0$$

$$s = 1$$

$$\theta = 90^\circ$$

2

$$c = 4/5$$

$$s = 3/5$$

$$c \equiv \cos \theta = L_x/L$$

$$s \equiv \sin \theta = L_y/L$$

$$(\text{global}) \quad \{F_f\} = [T]^T \{Q_f\}$$

$$\{F_f\}^1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \{F_f\}^2 = \begin{Bmatrix} 0 \\ 200 \\ 2e5 \\ 0 \\ 200 \\ -2e5 \end{Bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN}\cdot\text{mm} \\ \text{kN} \\ \text{kN} \\ \text{kN}\cdot\text{mm} \end{matrix}$$

$$[K]_{\text{global}} = [T]^T [k] [T]$$

$$[K]^1 = \begin{bmatrix} 0.192 & 0 & -480 & -0.192 & 0 & -480 \\ 0 & 120 & 0 & 0 & -120 & 0 \\ -480 & 0 & 1.6e6 & 480 & 0 & 0.8e6 \\ -0.192 & 0 & 480 & 0.192 & 0 & 480 \\ 0 & -120 & 0 & 0 & 120 & 0 \\ -480 & 0 & 0.8e6 & 480 & 0 & 1.6e6 \end{bmatrix}$$

$$[K]^2 = \begin{bmatrix} 76.87 & 57.51 & -288 & -76.87 & -57.51 & -288 \\ 57.51 & 43.32 & 384 & -57.51 & -43.32 & 384 \\ -288 & 384 & 1.6e6 & 288 & -384 & 0.8e6 \\ -76.87 & -57.51 & 288 & 76.87 & 57.51 & 288 \\ -57.51 & -43.32 & -384 & -57.51 & 43.32 & -384 \\ -288 & 384 & 0.8e6 & 288 & -384 & 1.6e6 \end{bmatrix}$$

4. Assembly
(code # method)

$$\{P_f\} = \begin{Bmatrix} F_{f4}^1 + F_{f1}^2 \\ F_{f5}^1 + F_{f2}^2 \\ F_{f6}^1 + F_{f3}^2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 200 \\ 2e5 \end{Bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN}\cdot\text{mm} \end{matrix}$$

$$[S] = \begin{bmatrix} K_{44}^1 + K_{11}^2 & K_{45}^1 + K_{12}^2 & K_{46}^1 + K_{13}^2 \\ K_{54}^1 + K_{21}^2 & K_{55}^1 + K_{22}^2 & K_{56}^1 + K_{23}^2 \\ K_{64}^1 + K_{31}^2 & K_{65}^1 + K_{32}^2 & K_{66}^1 + K_{33}^2 \end{bmatrix} = \begin{bmatrix} 77.06 & 57.51 & 192 \\ 57.51 & 163.32 & 384 \\ 192 & 384 & 3.2e6 \end{bmatrix}$$

5. Solve

$$\{P\} = \{P_f\} + [S]\{d\}$$

$$\{d\} = [S]^{-1} \{P - P_f\}$$

$$\{d\} = \begin{Bmatrix} 4.8224 \\ -2.7757 \\ -0.06246 \end{Bmatrix} \begin{matrix} \text{mm } d_1 \\ \text{mm } d_2 \\ \text{rad } d_3 \end{matrix}$$

Post-Process

6. Compute member-end forces : $\{F\} = \{F_f\} + [K]\{v\} \rightarrow$ compatibility w/ $\{d\}$
(global)

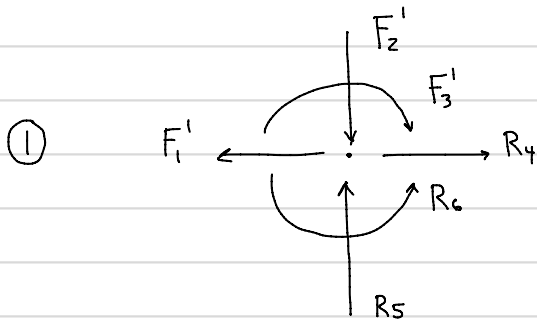
$$\{v\}^1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ d_1 \\ d_2 \\ d_3 \end{Bmatrix} \text{ fixed B.C.} \quad \{v\}^2 = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \text{ fixed B.C.}$$

$$\{F\}^1 = \begin{Bmatrix} 29.05 \\ 333.09 \\ -47.65 \\ 29.05 \\ -333.09 \\ -97.62 \end{Bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN}\cdot\text{m}^* \\ \text{kN} \\ \text{kN} \\ \text{kN}\cdot\text{m}^* \end{matrix} \quad \{F\}^2 = \begin{Bmatrix} 229.05 \\ 333.09 \\ 97.62 \\ -229.05 \\ 66.91 \\ -252.42 \end{Bmatrix} \begin{matrix} \text{kN} \\ \text{kN} \\ \text{kN}\cdot\text{m}^* \\ \text{kN} \\ \text{kN} \\ \text{kN}\cdot\text{m}^* \end{matrix} \text{ converted}$$

7. Support reactions
(joint equilibrium)

$$\{F\}^1 = \begin{Bmatrix} 29.05 & \text{kN} \\ 333.09 & \text{kN} \\ -47.65 & \text{kN}\cdot\text{m}^* \\ 29.05 & \text{kN} \\ -333.09 & \text{kN} \\ -97.62 & \text{kN}\cdot\text{m}^* \end{Bmatrix}$$

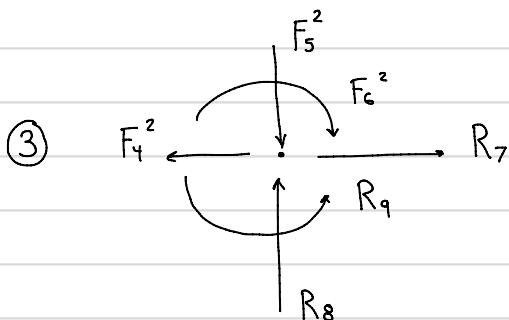
$$\{F\}^2 = \begin{Bmatrix} 229.05 & \text{kN} \\ 333.09 & \text{kN} \\ 97.62 & \text{kN}\cdot\text{m}^* \\ -229.05 & \text{kN} \\ 66.91 & \text{kN} \\ -252.42 & \text{kN}\cdot\text{m}^* \end{Bmatrix}$$



$$\sum F_x = 0 \quad R_4 = F_1' = 29.05 \text{ kN}$$

$$\sum F_y = 0 \quad R_5 = F_2' = 333.09 \text{ kN}$$

$$\sum M = 0 \quad R_6 = F_3' = -47.65 \text{ kN}\cdot\text{m} \quad \therefore \curvearrowright$$



$$\sum F_x = 0 \quad R_7 = F_4^2 = -229.05 \text{ kN} \quad \therefore \leftarrow$$

$$\sum F_y = 0 \quad R_8 = F_5^2 = 66.91 \text{ kN}$$

$$\sum M = 0 \quad R_9 = F_6^2 = -252.42 \text{ kN}\cdot\text{m} \quad \therefore \curvearrowright$$

* Check overall equilibrium *

8. Draw axial force, shear force, and bending moment diagrams

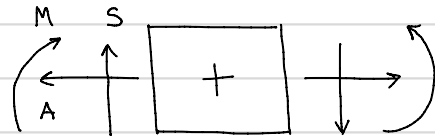
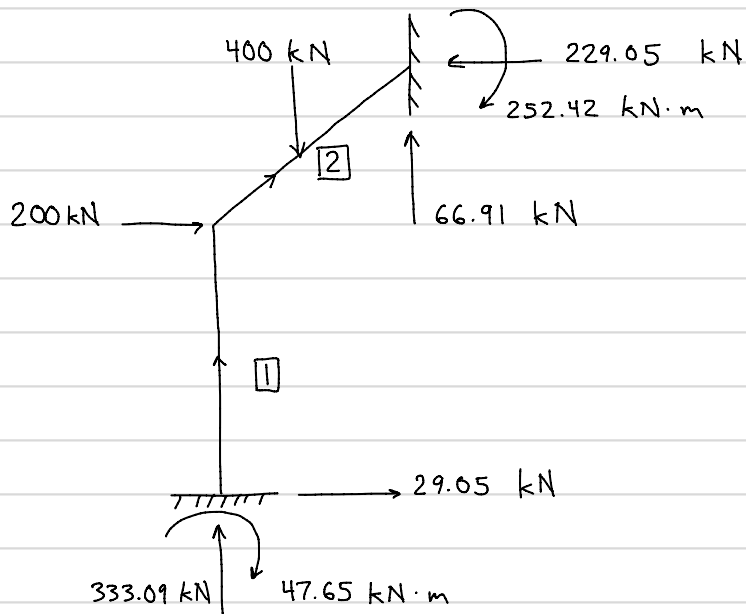
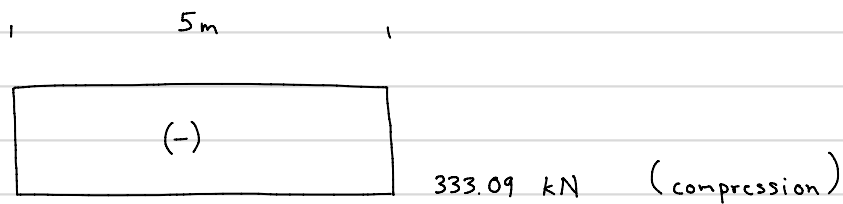


diagram positive sign convention

Member 1

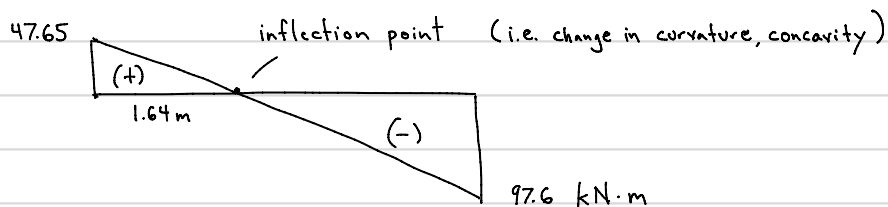
Axial



Shear

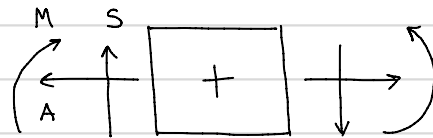
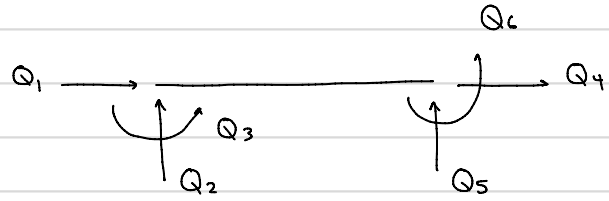


Moment

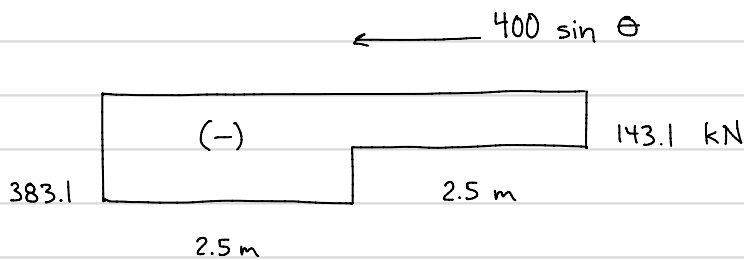


Member 2 (easier to work with Q) $\{Q\} = [T] \{F\}$

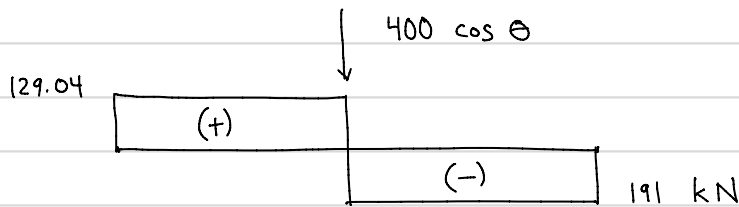
$$\{Q\}^2 = \begin{Bmatrix} 383.1 \\ 129.04 \\ 97.6 \\ -143.1 \\ 191 \\ -252.42 \end{Bmatrix} \begin{matrix} \text{kN} & \text{axial} \\ \text{kN} & \text{shear} \\ \text{kN}\cdot\text{m} & \text{moment} \\ \text{kN} & \\ \text{kN} & \\ \text{kN}\cdot\text{m} & \end{matrix}$$



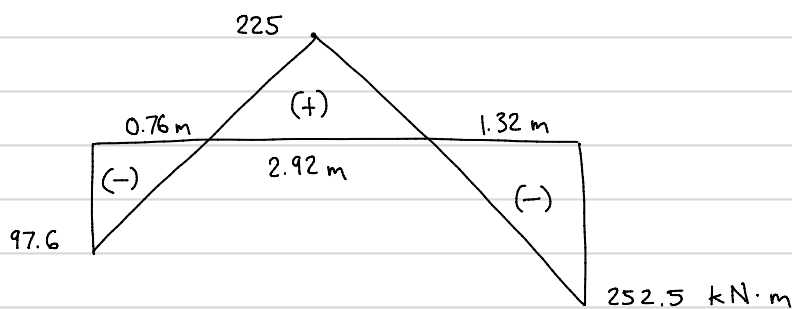
Axial



Shear



Moment

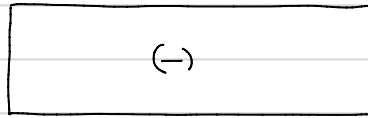


9. Calculate stresses

Member \square

Axial Stress

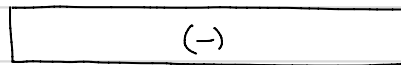
$$\sigma_a = \frac{\text{Force}}{\text{Area}}$$



11.1 MPa

$$\text{Area} = 3 \times 10^4 \text{ mm}^2$$

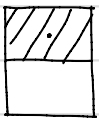
Shear Stress



1.45 MPa

$$\tau = \frac{VQ}{Ib}$$

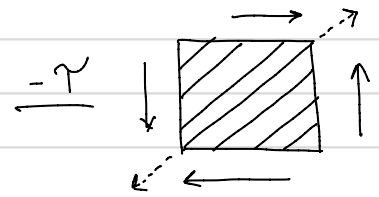
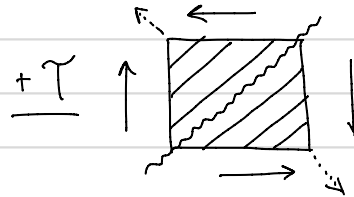
$$I = 1 \times 10^8 \text{ mm}^4$$



$$Q = (150 \cdot 100)(50) = 7.5 \times 10^5 \text{ mm}^3$$

* sign of shear stress important when:

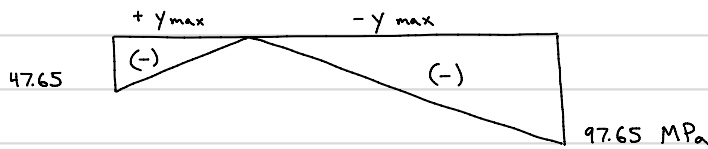
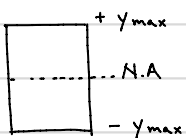
- combining w/ torsion effects (CE 525)
- anisotropic materials (e.g. fiber-composites)



Bending Stress

$$\sigma_b = \frac{-My}{I}$$

* $\sigma = \sigma_a + \sigma_b$. . . we need to consider worst case combination(s)



$$* \sigma_{\max} = -11.1 - 97.65 = -108.75 \text{ MPa (compression)}$$

$$* \tau_{\max} = 1.45 \text{ MPa}$$

$$\sigma_{\max} \text{ (tension)} = -11.1 + 97.65 = 86.55 \text{ MPa}$$

+ y_{max}

Member 2

Axial Stress

$$\sigma_a = \frac{P}{A}$$



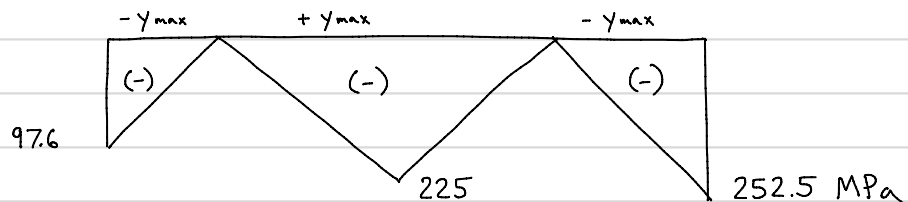
Shear Stress

$$\tau = \frac{VQ}{Ib}$$



Bending Stress

$$\sigma_b = -\frac{My}{I}$$



$$\sigma_{\max} = 257.27 \text{ MPa (compression)} \quad \tau_{\max} = 9.55 \text{ MPa}$$