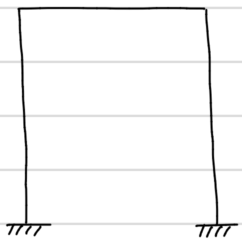
April 16, 2026
Matrix Displacement Method
Moment Releases
(Plane Frames)

Member Releases (plane frames)



rigid connections at joints

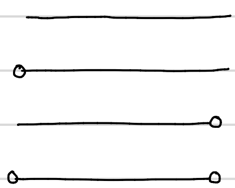
$\therefore$

member-end rotations equal to rotations of adjacent joint

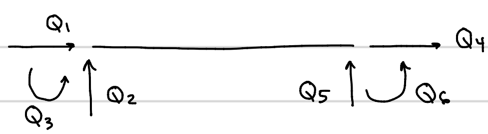
common practice in structural engineering to construct certain members with a hinge such that the moment is equal to zero "released"

In context of MDM, four members types (MT)

MT0 : no hinge
MT1 : hinge at beginning
MT2 : hinge at end
MT3 : hinge at both ends



recall $\{Q\} = \{Q_f\} + [k]\{u\}$



plane frame element

$$Q_1 = FA_b + \frac{EA}{L}(u_1 - u_4)$$
$$Q_2 = FS_b + \frac{EI}{L^3}(12u_2 + 6Lu_3 - 12u_5 + 6Lu_6)$$
$$Q_3 = FM_b + \frac{EI}{L^3}(6Lu_2 + 4L^2u_3 - 6Lu_5 + 2L^2u_6)$$
$$Q_4 = FA_e + \frac{EA}{L}(-u_1 + u_4)$$
$$Q_5 = FS_e + \frac{EI}{L^3}(-12u_2 - 6Lu_3 + 12u_5 - 6Lu_6)$$
$$Q_6 = FM_e + \frac{EI}{L^3}(6Lu_2 + 2L^2u_3 - 6Lu_5 + 4L^2u_6)$$

Above equations for <u>MT0</u>

MT1 :  i.e. $Q_3 = 0$

$$\therefore \quad u_3 = \frac{3}{2L}\left(-u_2 + u_5\right) - \frac{1}{2}u_6 - \frac{L}{4EI} FM_b$$

u_3 no longer independent (or degree of freedom), but function of end displacements u_2, u_5, and u_6

substituting the expression for u_3 back into member force-stiffness relations we arrive at :

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \underbrace{\begin{Bmatrix} FA_b \\ FS_b - \frac{3}{2L} FM_b \\ 0 \\ FA_e \\ FS_e + \frac{3}{2L} FM_b \\ FM_e - \frac{1}{2} FM_b \end{Bmatrix}}_{\{Q_f\}} + \frac{EI}{L^3} \underbrace{\begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 3 & 0 & 0 & -3 & 3L \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -3 & 0 & 0 & 3 & -3L \\ 0 & 3L & 0 & 0 & -3L & 3L^2 \end{bmatrix}}_{[k]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

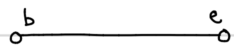
MT2 :  i.e. $Q_6 = 0$

$$\therefore \quad u_6 = \frac{3}{2L}\left(-u_2 + u_5\right) - \frac{1}{2}u_3 - \frac{L}{4EI} FM_e$$

substituting the expression for u_6 back into the member force-stiffness relations : $\{Q\} = \{Q_f\} + [k]\{u\}$
we arrive at :

$$\{Q_f\} = \begin{Bmatrix} FA_b \\ FS_b - \frac{3}{2L} FM_e \\ FM_b - \frac{1}{2} FM_e \\ FA_e \\ FS_e + \frac{3}{2L} FM_e \\ 0 \end{Bmatrix}, \quad [k] = \frac{EI}{L^3} \begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 3 & 3L & 0 & -3 & 0 \\ 0 & 3L & 3L^2 & 0 & -3L & 0 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -3 & -3L & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

MT3 :



i.e. $Q_3 = Q_6 = 0$

$$u_3 = \frac{1}{L}\left(-u_2 + u_5\right) - \frac{L}{6EI}\left(2FM_b - FM_e\right)$$

$$u_6 = \frac{1}{L}\left(-u_2 + u_5\right) - \frac{L}{6EI}\left(2FM_e - FM_b\right)$$

substituting the expressions for u_3, u_6 back into member force-stiffness relations we arrive at :

$$\{Q_f\} = \begin{Bmatrix} FA_b \\ FS_b - \frac{1}{L}\left(FM_b + FM_e\right) \\ 0 \\ FA_e \\ FS_e + \frac{1}{L}\left(FM_b + FM_e\right) \\ 0 \end{Bmatrix}, \quad [k] = \frac{EA}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

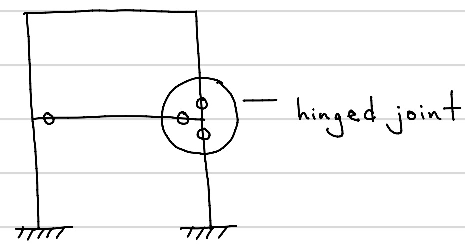
(same terms as planar truss)

Notes : * although the number of independent member coordinates is reduced, orders of modified $\{Q_f\}$ and $[k]$ is maintained at 6×1 and 6×6 respectively, eliminating need to modify $[T]$ for plane frames providing an efficient way to implement in a computer program

** since the member-end displacement vector $\{u\}$ is determined from compatibility of $\{v\}$ with joint displacements $\{d\}$, corresponding member-end rotations at a hinge will be ZERO and must be calculated using the appropriate equations for the dependent DOFs,

e.g. MT1 : $u_3 = \frac{3}{2L}\left(-u_2 + u_5\right) - \frac{1}{2}u_6 - \frac{L}{4EI}FM_b$

$\boxed{\text{Hinged joints}}$ - if all members meeting at a joint are connected by a hinge, that joint is a hinged joint

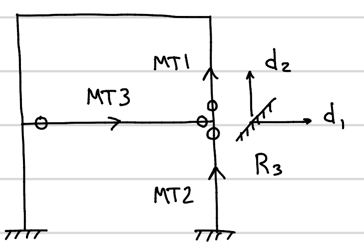


since hinged joints cannot transfer any moment and free to rotate, rotational stiffnesses are zero. Thus including rotational DOFs causes [S] to become singular, i.e. determinant = 0 ∴ non-invertible

Two solutions in order to model this:

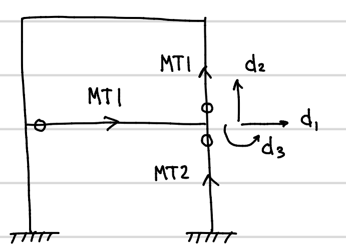
1. eliminate rotational DOFs from hinged joint

since hinged joint is not subjected to moments, rotations are zero (though released ends of members connected to such a joint can and do, rotate)



imaginary clamp applied with corresponding reaction R_3 (since hinged joints are not subjected to any external moments and moments at ends of all members are equal to zero, $R_3 = 0$)

2. overcome singularity in [S] by rigidly connecting one and only one member



since no external moments applied to joint and moments at ends of all but one of the members meeting at joint are zero, the moment at the end of the one member that is rigidly connected to the joint equals zero due to moment equilibrium

Both approaches yield identical results, but the first approach can provide a more computationally efficient method since there are less DOFs