Matrix Displacement Method (MDM) Summary

Uniaxial: One (1) *structural-level* Degree of freedom (DOF): d_x – axial displacement

$$\{P\}=\{P_f\}+[S]\{d\}$$

Member end forces {Q}: 2 (axial forces 1,2) $\{Q\}=\{Q_f\}+[k]\{u\}$

Member stiffness matrix: [k] 2 x 2 $[k]=\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Solution Steps

1. Label all members, joints, DOFs (d_x), and support reactions.
2. Create the *structural* force vector {P}
3. Determine all *member* stiffness matrices [k], and fixed end force vectors $\{Q_f\}$
4. Use the Code Number method to assemble the *structural* stiffness matrix [S] and fixed end force vector $\{P_f\}$
5. Solve for the DOFs {d} using $\{d\} = [S]^{-1}\{P-P_f\}$
6. Computer *member end* forces using $\{Q\} = \{Q_f\} + [k]\{u\}$ and compatibility of {d}-{u}
7. Determine support **reactions** from joint equilibrium
8. Draw *axial force* diagrams, determine axial **stress** from $\sigma = \frac{N}{A}$

2D Truss: Two (2) *structural-level* DOFs: d_X, d_Y – X,Y displacements

$$\{P\}=[S]\{d\}$$

<u>Member end forces:</u>

Local - {Q}: 4 (axial 1,3 & transverse forces 2,4) $\{Q\}=[k]\{u\}$

Global - {F}: 4 (X 1,3 & Y 2,4 forces) $\{F\}=[K]\{v\}$

{Q} = [T]{F}, {u} = [T]{v}

$$[T]=\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

<u>Member stiffness matrix:</u>

Local - [k] 4 x 4

$$[k]=\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Global - $[K]=[T]^T[k][T]$

$$[K]=\frac{EA}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

Solution Steps

1. Label all members (w/ local coordinate axes), joints, DOFs (d_X,d_Y), and support reactions.
2. Create the *structural* force vector {P}
3. Determine all *member* stiffness matrices in ***local*** coordinates [k], transformation matrices [T], and ***global*** stiffness matrices via $[K] = [T]^T[k][T]$.
4. Use the Code Number method to assemble the *structural* stiffness matrix [S]
5. Solve for the DOFs {d} using $\{d\} = [S]^{-1}\{P\}$
6. Computer *member end* forces in ***global*** coordinates using {F} = [K]{v} and compatibility of {d}-{v}
7. Calculate axial "**bar**" forces in each member:
   Option 1: {Q} = [T]{F} and report Q_3 since (+) is tension and (-) is compression
   Option 2: Use Pythagorean Theorem with F_1, F_2 to determine local axial force
8. Determine support **reactions** from joint equilibrium
9. Determine axial **stress** from $\sigma=\frac{N}{A}$

3D Spatial Truss: Three (3) *structural-level* DOFs: d_X, d_Y, d_Z – X,Y,Z displacements

$$\{P\}=[S]\{d\}$$

<u>Member end forces:</u>

Local - {Q}: 2 (axial forces 1,2) $\{Q\}=[k]\{u\}$

Global - {F}: 6 (X 1,4 & Y 2,5 & Z 3,6 forces) $\{F\}=[K]\{v\}$

{Q} = [T]{F}, {u} = [T]{v}

$$[T]=\begin{bmatrix} \cos\theta_X & \cos\theta_Y & \cos\theta_Z & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_X & \cos\theta_Y & \cos\theta_Z \end{bmatrix}$$

<u>Member stiffness matrix:</u>

Local - [k] 2 x 2

$$[k]=\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Global - $[K]=[T]^T[k][T]$

$$[K]=\frac{EA}{L}\begin{bmatrix} c^2\theta_X & c\theta_X c\theta_Y & c\theta_X c\theta_Z & -c^2\theta_X & -c\theta_X c\theta_Y & -c\theta_X c\theta_Z \\ - & c^2\theta_Y & c\theta_Y c\theta_Z & -c\theta_X c\theta_Y & -c^2\theta_Y & -c\theta_Y c\theta_Z \\ - & - & c^2\theta_Z & -c\theta_X c\theta_Z & -c\theta_Y c\theta_Z & -c^2\theta_Z \\ - & - & - & c^2\theta_X & c\theta_X c\theta_Y & c\theta_X c\theta_Z \\ - & - & - & - & c^2\theta_Y & c\theta_Y c\theta_Z \\ - & - & - & - & - & c^2\theta_Z \end{bmatrix}$$

<u>Solution Steps</u>

1. Label all members (w/ local coordinate axes), joints, DOFs (d_X,d_Y,d_Z), and support reactions.
2. Create the *structural* force vector {P}
3. Determine all *member* stiffness matrices in ***local*** coordinates [k], transformation matrices [T], and ***global*** stiffness matrices via $[K] = [T]^T[k][T]$.
4. Use the Code Number method to assemble the *structural* stiffness matrix [S]
5. Solve for the DOFs {d} using $\{d\} = [S]^{-1}\{P\}$
6. Computer *member end* forces in ***global*** coordinates using {F} = [K]{v} and compatibility of {d}-{v}
7. Calculate axial "**bar**" forces in each member:
   Option 1: {Q} = [T]{F} and report Q_2 since (+) is tension and (-) is compression
   Option 2: Use Pythagorean Theorem with F_1, F_2, F_3 to determine local axial force
8. Determine support **reactions** from joint equilibrium
9. Determine axial **stress** from $\sigma=\frac{N}{A}$

Beam: Two (2) *structural-level* DOFs: d_v, d_θ – transverse displacement, rotation

$$\{P\}=\{P_f\}+[S]\{d\}$$

Member end forces {Q}: 4 (transverse forces 1,3 – moments 2,4)

$$\{Q\}=\{Q_f\}+[k]\{u\}$$

Member stiffness matrix (BEBT): [k] 4 x 4

$$[k]=\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Member stiffness matrix (Timoshenko): [k] 4 x 4

$$[k]=\frac{EI}{L^3(1+\beta_S)}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & L^2(4+\beta_S) & -6L & L^2(2-\beta_S) \\ -12 & -6L & 12 & -6L \\ 6L & L^2(2-\beta_S) & -6L & L^2(4+\beta_S) \end{bmatrix} \qquad \beta_S=\frac{12EI\,f_S}{GAL^2} \quad f_S=6/5 \text{ rectangle}$$

Solution Steps

1. Label all members, joints, DOFs (d_v,d_θ), and support reactions.
2. Create the *structural* force vector {P}
3. Determine all *member* stiffness matrices [k], and fixed end force vectors {Q_f}
4. Use the Code Number method to assemble the *structural* stiffness matrix [S] and fixed end force vector {P_f}
5. Solve for the DOFs {d} using $\{d\} = [S]^{-1}\{P-P_f\}$
6. Computer *member end* forces using $\{Q\} = \{Q_f\} + [k]\{u\}$ and compatibility of {d}-{u}
7. Determine support **reactions** from joint equilibrium
8. Draw *shear force* and *bending moment* diagrams
9. Determine shear and bending **stresses** from $\tau=\frac{VQ}{Ib}$, $\sigma=-\frac{My}{I}$

2D Frame: Three (3) *structural-level* DOFs: d_X, d_Y, d_θ – XY translations, rotation

$$\{P\}=\{P_f\}+[S]\{d\}$$

Member end forces:

Local - {Q}: 6 (axial forces 1,4 - transverse forces 2,5 – moments 3,6) $\{Q\}=\{Q_f\}+[k]\{u\}$

Global - {F}: 6 (X forces 1,4 - Y forces 2,5 – moments 3,6) $\{F\}=\{F_f\}+[K]\{v\}$

{Q} = [T]{F}, {u} = [T]{v}

$$[T]=\begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Member stiffness matrix:

Local – [k] 6 x 6

$$[k]=\begin{bmatrix} EA/L & 0 & 0 & -EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -EA/L & 0 & 0 & EA/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix}$$

Solution Steps

1. Label all members, joints, DOFs (d_X,d_Y,d_θ), and support reactions.
2. Create the *structural* force vector {P}
3. Determine all *member* stiffness matrices [k] and fixed end force vectors $\{Q_f\}$ in ***local*** coordinates, transformation matrices [T], and ***global*** stiffness matrices and global force vectors via $[K] = [T]^T[k][T]$, $\{F_f\} = [T]^T\{Q_f\}$.
4. Use the Code Number method to assemble the *structural* stiffness matrix [S] and fixed end force vector $\{P_f\}$
5. Solve for the DOFs {d} using $\{d\} = [S]^{-1}\{P-P_f\}$
6. Computer *member end* forces in ***global*** coordinates using $\{F\} = \{F_f\} + [K]\{v\}$ and compatibility of {d}-{v}
7. Determine support **reactions** from joint equilibrium
8. Draw *axial*, *shear force* and *bending moment* diagrams: {Q} = [T]{F}
9. Determine shear and normal (axial + bending) **stresses** from $\tau=\frac{VQ}{Ib}$, $\sigma_a=\frac{N}{A}$, $\sigma_b=-\frac{My}{I}$

Grids: Three (3) *structural-level* DOFs: d_Y, $d_{\theta X}$, $d_{\theta Z}$ –Y translation, XZ rotation

$$\{P\}=\{P_f\}+[S]\{d\}$$

Member end forces:

Local - {Q}: 6 (transverse forces 1,4 – moments 2-3, 5-6) $\{Q\}=\{Q_f\}+[k]\{u\}$

Global - {F}: 6 (Y forces 1,4 – XZ moments 2-3, 5-6) $\{F\}=\{F_f\}+[K]\{v\}$

{Q} = [T]{F}, {u} = [T]{v}

$$[T]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & s & 0 & 0 & 0 \\ 0 & -s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & s \\ 0 & 0 & 0 & 0 & -s & c \end{bmatrix}$$

Member stiffness matrix:

Local – [k] 6 x 6

$$[k]=\begin{bmatrix} 12EI/L^3 & 0 & 6EI/L^2 & -12EI/L^3 & 0 & 6EI/L^2 \\ 0 & GJ/L & 0 & 0 & -GJ/L & 0 \\ 6EI/L^2 & 0 & 4EI/L & -6EI/L^2 & 0 & 2EI/L \\ -12EI/L^3 & 0 & -6EI/L^2 & 12EI/L^3 & 0 & -6EI/L^2 \\ 0 & -GJ/L & 0 & 0 & GJ/L & 0 \\ 6EI/L^2 & 0 & 2EI/L & -6EI/L^2 & 0 & 4EI/L \end{bmatrix}$$

Solution Steps

1. Label all members, joints, DOFs (d_Y, $d_{\theta X}$, $d_{\theta Z}$), and support reactions.
2. Create the *structural* force vector {P}
3. Determine all *member* stiffness matrices [k] and fixed end force vectors $\{Q_f\}$ in ***local*** coordinates, transformation matrices [T], and ***global*** stiffness matrices and global force vectors via $[K] = [T]^T[k][T]$, $\{F_f\} = [T]^T\{Q_f\}$.
4. Use the Code Number method to assemble the *structural* stiffness matrix [S] and fixed end force vector $\{P_f\}$
5. Solve for the DOFs {d} using $\{d\} = [S]^{-1}\{P-P_f\}$
6. Computer *member end* forces in ***global*** coordinates using $\{F\} = \{F_f\} + [K]\{v\}$ and compatibility of {d}-{v}
7. Determine support **reactions** from joint equilibrium
8. Draw *torsion*, *shear force* and *bending moment* diagrams: {Q} = [T]{F}
9. Determine shear (torsion + force) and normal **stresses** from $\tau_T=\frac{T\rho}{J}$, $\tau_S=\frac{VQ}{Ib}$, $\sigma_b=-\frac{My}{I}$

Space Frames: Six (6) *structural-level* DOFs: $d_{X,Y,Z}$, $d_{\theta X,\theta Y,\theta Z}$ – X,Y,Z translations/rotations

$$\{P\}=\{P_f\}+[S]\{d\}$$

<u>Member end forces:</u>

Local - {Q}: 12 (transverse forces 1-3, 7-9 – moments 4-6, 10-12) $\{Q\}=\{Q_f\}+[k]\{u\}$

Global - {F}: 12 (XYZ forces 1-3, 7-9 – XYZ moments 4-6, 10-12) $\{F\}=\{F_f\}+[K]\{v\}$

<u>Member stiffness matrix:</u> *Local* – [k] 12 x 12

$$[k]=\begin{bmatrix}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{-EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & 0 & 0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{-GJ}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
\frac{-EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} \\
0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix}$$

<u>Transformations:</u> {Q} = [T]{F}, {u} = [T]{v}

$$\underset{12x12}{[T]}=\begin{bmatrix}
[r] & [0] & [0] & [0] \\
[0] & [r] & [0] & [0] \\
[0] & [0] & [r] & [0] \\
[0] & [0] & [0] & [r]
\end{bmatrix}$$

$\underset{3x3}{[0]}$ null matrix $\underset{3x3}{[r]}$ rotation matrix

For arbitrarily oriented members (*except vertical*):

$$[r]=\begin{bmatrix} r_{xX} & r_{xY} & r_{xZ} \\ \left(-r_{xX}r_{xY}\cos\psi - r_{xZ}\sin\psi\right)/\left|\bar{Z}\right| & \left|\bar{Z}\right|\cos\psi & \left(-r_{xY}r_{xZ}\cos\psi + r_{xX}\sin\psi\right)/\left|\bar{Z}\right| \\ \left(r_{xX}r_{xY}\sin\psi - r_{xZ}\cos\psi\right)/\left|\bar{Z}\right| & -\left|\bar{Z}\right|\sin\psi & \left(r_{xY}r_{xZ}\sin\psi + r_{xX}\cos\psi\right)/\left|\bar{Z}\right| \end{bmatrix}$$

where $r_{xX}=\dfrac{X_e-X_b}{L}$, $r_{xY}=\dfrac{Y_e-Y_b}{L}$, $r_{xZ}=\dfrac{Z_e-Z_b}{L}$ and $L=\sqrt{\left(X_e-X_b\right)^2+\left(Y_e-Y_b\right)^2+\left(Z_e-Z_b\right)^2}$

$\left|\bar{Z}\right|\equiv\sqrt{r_{xZ}^2+r_{xX}^2}$ and $\psi\equiv$ roll angle

**For *vertical* members: $[r]=\begin{bmatrix} 0 & r_{xY} & 0 \\ -r_{xY}\cos\psi & 0 & \sin\psi \\ r_{xY}\sin\psi & 0 & \cos\psi \end{bmatrix}$

Solution Steps

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4. Use the Code Number method to assemble the *structural* stiffness matrix [S] and fixed end force vector {P_f}
5. Solve for the DOFs {d} using $\{d\} = [S]^{-1}\{P-P_f\}$
6. Computer *member end* forces in ***global*** coordinates using $\{F\} = \{F_f\} + [K]\{v\}$ and compatibility of {d}-{v}
7. Determine support **reactions** from joint equilibrium
8. Draw *axial, torsion*, *shear force* and *bending moment* diagrams: {Q} = [T]{F}
9. Determine ***normal*** (axial + bending) **stresses** from $\sigma_a=\dfrac{N}{A}$, $\sigma_b=-\dfrac{My}{I}$
10. Determine ***shear*** (torsion + force) **stresses** from $\tau_T=\dfrac{T\rho}{J}$, $\tau_S=\dfrac{VQ}{Ib}$